A survey of (BIBO) stability and (proper) stabilization of multidimensional input/output systems

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1 Introduction

Stability and stabilization theory is a part of control theory and usually involves the following

Ingredients. [22, p. 60] 1. Stability: Select the class of admissible systems and define and characterize the stable systems in this class and, in particular, check BIBO (bounded input/bounded output) stability of stable systems. 2. Stabilizability: Which admissible systems can be stabilized by output feedback? 3. Stabilization: Construct a stabilizing compensator for a given stabilizable system. 4. Parametrization: Classify or construct all stabilizing compensators for a given stabilizable system.

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In this survey we discuss these problems for continuous and discrete multidimensional input/output (IO) behaviors which are described by linear systems of complex partial differential equations on $\mathbb{R}^r$ resp. difference equations on $\mathbb{N}^r$ with constant coefficients, the signals being taken from various function spaces, in particular from those of polynomial-exponential functions. After an introduction into multidimensional behavioral systems theory we give a survey without detailed proofs, but sufficient motivations of [26] and of our talks at the conferences D2 and D3 of the Gröbner semester 2006 in Linz and also present new results on BIBO stability. Complete proofs of the new results on BIBO stability and proper stabilization will appear in [38] and [39].

J. W. Polderman and J. C. Willems in [31, section 10.8] and P. Rocha [36] suggest to renounce the input/output structure and output feedback in favor of more general behavior interconnections. The first stabilization results for multidimensional systems in this generality are due to S. Shankar [41, 42].

In contrast to our approach most multidimensional stability and stabilization papers use a commutative integral domain $S$ of ‘SISO stable plants’ and describe the admissible systems, often called plants, by a transfer operator or input/output map which is a matrix with coefficients in the quotient field of $S$. These systems are called structurally [16] or internally [33] stable if their transfer matrix has entries in $S$. In the one-dimensional case this approach to stabilization theory is due to C. A. Desoer, V. Kucera, M. Vidyasagar, D. C. Youla, and their coworkers. Among the well-known contributors to multidimensional stability and stabilization are M. Bisiacco [2], N. K. Bose [6, 9, 10, 7, 8], D. E. Dudgeon [11], E. Fornasini [13], J. P. Guiver [16], E. I. Jury [21], Z. Lin [22, 23], G. Marchesini [13], R. M. Mersereau [11], A. Quadrat [32, 33, 34], H. C. Reddy [35], S. Shankar [41, 42, 43], V. R. Sule [45], J. Wood [51], L. Xu [52], J. Q. Ying [53], E. Zerz [54] et al. We refer the reader especially to the interesting surveys [6, 7, 8] by N. K. Bose and [21] by E. I. Jury with detailed descriptions of the results, problems, history and further important contributors of the subject and with comprehensive lists of references.

An IO behavior $B$ gives rise to its autonomous part $B^0$, its transfer matrix $H$ and to the characteristic varieties $\text{char}(B) \subseteq \text{char}(B^0)$ of $B$ and of $B^0$. The entries of the transfer matrix are complex rational functions in $r$ indeterminates $s_\rho$, i.e., are contained in the quotient field $\mathbb{C}(s)$ of the polynomial algebra $A := \mathbb{C}[s] = \mathbb{C}[s_1, \ldots, s_r]$. In the one-dimensional theory the elements of $\text{char}(B^0)$ are called the poles, modes, characteristic values or natural frequencies of the system.

Stability and stabilization of an IO system are defined with respect to a disjoint decomposition $\mathbb{C}^r = \Lambda_1 \sqcup \Lambda_2$ of the complex space into a stable region $\Lambda_1$ and an unstable region $\Lambda_2$, the standard continuous resp. discrete cases being

$$
\Lambda_{2,\text{cont}} = \mathbb{C}^r_{\geq 0}, \quad \mathbb{C}^r_{\geq 0} := \{z \in \mathbb{C}; \Re(z) \geq 0\} \text{ resp. } \Lambda_{2,\text{dis}} = \{z \in \mathbb{C}; |z| \geq 1\}^r. \quad (1.1)
$$

A polynomial $t \in A$ is called stable if its zeros lie in the stable region, i.e., if it belongs to $T := \{t \in A; \forall \lambda \in \Lambda_2 : t(\lambda) \neq 0\} [45]$. A rational function is called stable if its poles lie in the stable region, i.e., if it belongs to the quotient ring $A_T \subseteq \mathbb{C}(s)$ of ‘SISO stable plants’. An IO system $B$ is called stable if the characteristic variety of its autonomous part is contained in the stable region or, equivalently, if all polynomial-exponential trajectories in $B^0$ involve exponents in the stable region only (Theorem 6.2, 1.(i),(ii)). In [51] this is called a CV-condition and used to define stable autonomous
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systems. Stability of \( B \) in this sense is equivalent with the stability of the transfer matrix and an additional condition (Theorem 6.2, 1.(iii)). A one-dimensional IO system is stable with respect to (1.1) if and only if its autonomous part is asymptotically stable. A stable IO system is \textit{input/output stable} in the sense that the transfer matrix acts as operator on interesting classes of inputs and generates outputs of the same type as shown in parts 2, 3 and 4 of Theorem 6.2, but BIBO stability does not hold in general.

A rational function in \( \mathbb{C}(s) \) is called \textit{proper} if it is a formal and then automatically a convergent power series in \( s_1^{-1}, \ldots, s_r^{-1} \). Properness of (the entries of) the transfer matrix \( H \) is not generally assumed in [26] or in the present survey since standard transfer functions like \( (s_1 - s_2)^{-1}, (s_2^2 - s_3^2)^{-1} \) (wave equation) or \( (s_1 - s_2^2)^{-1} \) (heat equation) are not proper. Properness of the transfer function or matrix is, however, usually assumed in the literature and has often the desirable consequence that the transfer matrix can be considered as an operator or input/output map on large classes of inputs and that a proper stable transfer matrix is also BIBO stable. In part of the engineering literature on stability (compare [21]) stability of a proper rational function is defined as BIBO stability. We prove in Theorem 10.2 in the standard discrete case of equation (1.1) that proper stable rational functions in our sense coincide precisely with the structurally stable rational functions of the literature [16, 22] and are BIBO stable in particular. In the continuous case of equation (1.1) stable polynomials in our sense are called strict Hurwitz in the literature [21, Definition 5 on p. 140]. E. I. Jury conjectured BIBO stability of the inverse of a bivariate very strict Hurwitz polynomial [21, Remark on p. 143]. We prove this conjecture in Theorem 11.7. The methods of its proof will also be applicable to higher dimensional cases.

An IO system \( B \) is called \textit{stabilizable} if there is a compensator IO system \( B' \) such that the feedback system of \( B \) and \( B' \) is well-posed [49] and stable, and \( B' \) is then called a \textit{stabilizing compensator}. In Theorem 8.1 we characterize stabilizability of \( B \) and construct all stabilizing compensators (parametrization). In particular, a controllable IO system is stabilizable if and only if its transfer matrix is stabilizable in the usual sense [33]. The famous prototype of the parametrization in the one-dimensional case is that of V. Kucera, D. C. Youla, J. J.Bongiorno and H. A. Jabr and is exposed by M. Vidyasagar in [49, chapter 5]. The proofs employ localization after P. Gabriel as a new mathematical tool in systems theory which is described in B. Stenström’s book [44], in [26] and in section 7. At no place the results and proofs need or employ the so-called fractional representation approach, i.e., matrix fraction descriptions of the transfer matrix of various kinds and the, sometimes long [49, 33, 34], ensuing matrix computations, and seem thus simpler and of interest even in the one-dimensional case. The localization technique also avoids the difficulties in [51] with the lack of ideal-convexity [43] of the unstable regions \( \Lambda_2 \). Ideal-convexity is characterized by the coincidence of Gabriel localization with the standard localization functor \( M \mapsto M_T \) on \( A \)-modules \( M \) (Theorem 7.2). Algorithm 8.2 discusses tests for stabilizability and the construction of all stabilizing compensators. There remain open problems, however. The difficulties are related to those pointed out in [52].

An IO system is called \textit{proper stabilizable} if it is stabilizable with the additional property that the transfer matrix of the resulting feedback system is proper (and, of course, stable). In Theorem 9.6 we characterize proper stabilizability and in Algorithm 9.7 we describe an algorithmic test for proper stabilizability and construct all proper
stabilizing compensators. However, as in Algorithm 8.2, there remain open problems. In the one-dimensional case proper stabilization of transfer matrices is discussed in [49, chapter 5] and of (Rosenbrock) systems in [48, chapter 6]. The table of contents exhibits the plan of the paper in detail.

2 Input/output systems

In engineering mathematics input/output (IO) systems are often pictorially represented as

![IO System](image)

where \( u \) resp. \( y \) are the input resp. the output signal of the system and are, in most cases and also in this paper, functions into finite-dimensional complex spaces \( \mathbb{C}^m \) resp. \( \mathbb{C}^p \). In colloquial terms, the system produces the output \( y \) from the input \( u \). In this paper the system is given as a multidimensional behavior which will be defined below, hence the letter \( B \). The attribute plant refers to a primary system whose behavior is to be steered or controlled.

In general, the assignment \( u \mapsto y \) is not a map (=operator, functional), but the system may, for instance, autonomously produce a non-zero output \( y \) from a zero input \( u = 0 \). If this assignment is actually a map \( u \mapsto y = H(u) \) then \( H \) is called an input/output map or transfer operator.

In context with stability and stabilization of IO systems by output feedback one has

1. to describe the admissible systems in exact mathematical form, in particular the spaces of admissible signals
2. and to characterize the stable systems among the admissible ones.

As usual we use output feedback systems as in Figure 2.2 with the following interpretation: The plant \( B_1 \) is controlled by a controller or compensator system \( B_2 \) which will often be a computer. For \( i = 1, 2 \) we assume that the input \( u_i \) of \( B_i \) and the output \( y_{3-i} \) of \( B_{3-i} \) have the same dimension and can therefore be added. The output \( y_1 \) of the plant and a chosen input \( u_2 \) of the controller are combined, i.e., summed up, and produce the output \( y_2 \) of the controller. This, in turn, is combined with the input \( u_1 \) of
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the plant and then produces its output $y_1$. The resulting total system is

$$B := \text{feedback}(B_1, B_2)$$

with input $\begin{pmatrix} u_2 \\ u_1 \end{pmatrix}$ and output $\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$ (2.1)

of the same dimension $p + m$. Mathematically the roles of $B_1$ and $B_2$ can be inter-
changed, but in technical realizations there is, of course and in general, a big difference
between the primary plant and its controller which is used to steer the plant.

Once the admissible systems and their stability are defined as will be done belo
we can also define the stabilization of a system.

Definition 2.1 An IO systems $B_1$ is called stabilizable if there is another IO system $B_2$ such that the feedback system $B := \text{feedback}(B_1, B_2)$ is stable. Then $B_2$ is called a stabilizing compensator or controller of $B_1$. The construction of all stabilizing compensators of one given plant $B_1$ is usually described by the term parametrization. If a stable compensator can be found one talks about strong stabilization.

3 Multidimensional signal modules

We are going to discuss systems governed by linear partial differential resp. difference equations with constant coefficients and talk, as usual in engineering mathematics, about the continuous resp. the discrete case. We start with the description of the continuous signals.

Consider the simple linear differential system

$$\dot{y} = Ay, \quad A \in \mathbb{R}^{n \times n} \quad \text{where} \quad y : \mathbb{R} \to \mathbb{R}, \; t \mapsto y(t),$$

with the solution $y(t) = e^{tA}y(0)$. It is a standard result that the stability behavior of this system for $t \to \infty$ is determined by the complex eigenvalues of $A$. In the sequel we therefore consider complex-valued functions $y$.

Let $r \geq 1$. We consider (multidimensional) signals of $r$ independent variables.
Definition 3.1 (Continuous multidimensional signals) Let

\[ A := \mathbb{C}[s] := \mathbb{C}[s_1, \ldots, s_r] \]

denote the complex polynomial algebra in \( r \) indeterminates \( s_1, \ldots, s_r \). The letter \( s \) is chosen since the indeterminates \( s_\rho \) act as shifts on discrete signals (see below). Another reason why the letter \( s \) is common in this setting is, that most engineers in systems theory introduce polynomial and rational matrices by means of the Laplace transform where \( s \) is the usual notation for the independent complex variable. Consider the function spaces

\[ C^\infty(\mathbb{R}^r) = \{ \text{arbitrarily often differentiable functions} \} \]
\[ \subset D' := D'(\mathbb{R}^r) = \{ \text{distributions} \} \]

of arbitrarily often differentiable, complex-valued functions

\[ y = y(t_1, \ldots, t_r), \ t = (t_1, \ldots, t_r) \in \mathbb{R}^r, \]

resp. of distributions. These spaces become \( \mathbb{C}[s] \)-modules via the action

\[ s_\rho \circ y := \partial y / \partial t_\rho \]

by partial differentiation.

In engineering practice \( C^\infty \)-signals are too restrictive since piecewise continuous signals appear frequently, for instance the one-dimensional Heaviside step function

\[ Y(t) := \begin{cases} 1 & \text{if } t \geq 0 \\ 0 & \text{if } t < 0 \end{cases} \]

The space of distributions contains these and is closed under differentiation, i.e., a \( \mathbb{C}[s] \)-module. These two properties are the reason that distributions, in particular Dirac’s \( \delta = Y' \) and its derivatives, play such an important part in engineering mathematics.

For the purposes of the present paper the space of polynomial-exponential functions is of particular importance.

Lemma and Definition 3.2 (Continuous polynomial-exponential functions) [28, Theorem 6.6] A distribution \( y \) is called locally finite if it generates a \( \mathbb{C} \)-finite dimensional cyclic module \( \mathbb{C}[s] \circ y \) consisting of all derivatives of \( y \) and their \( \mathbb{C} \)-linear combinations. The \( \mathbb{C}[s] \)-submodule of all locally finite distributions is the space of polynomial exponential functions

\[ \mathcal{F}_{\text{cont}} := D'_{\text{lf}} := \{ y \in D' (\mathbb{R}^r); \ \dim_{\mathbb{C}} (\mathbb{C}[s] \circ y) < \infty \} = \oplus_{\lambda \in \mathbb{C}^r} \mathbb{C}[t] e^{\lambda \bullet t}, \]

\[ \lambda = (\lambda_1, \ldots, \lambda_r) \in \mathbb{C}^r, \ t = (t_1, \ldots, t_r) \in \mathbb{R}^r, \]

\[ \lambda \bullet t := \lambda_1 t_1 + \ldots + \lambda_r t_r. \]  

(3.1)

In particular the locally finite distributions are entire functions.
The preceding \( \mathbb{C}[s] \)-modules are \textit{injective cogenerators}, see e.g. [37, p. 65ff and p. 79ff] for a broader explanation of these concepts. This fact is of paramount significance for the solution of linear systems of partial differential equations with constant coefficients in the function modules just introduced. To discuss this property in more detail we first introduce some standard objects from polynomial algebra.

Let \( \text{Max}(A) \) denote the set of all maximal ideals of the commutative ring \( A \). Hilbert’s Nullstellensatz implies the bijection

\[
\mathbb{C}^r \cong \text{Max}(\mathbb{C}[s]), \quad \lambda \mapsto m_\lambda,
\]

where

\[
m_\lambda := \mathbb{C}[s] \langle s_1 - \lambda_1, \ldots, s_r - \lambda_r \rangle = \{ t \in A; t(\lambda) = 0 \}.
\]

The \( \mathbb{C} \)-one-dimensional factor modules

\[
A/m_\lambda \cong \mathbb{C}, \quad f \mapsto f(\lambda), \quad \lambda \in \mathbb{C}^r,
\]

constitute a representative system of the \textit{irreducible} or \textit{simple} \( \mathbb{C}[s] \)-modules. Each \( \lambda \in \mathbb{C}^r \) gives also rise to the \textit{local ring} of rational functions defined in \( \lambda \), i.e., to

\[
A/m_\lambda = \{ f, t \in \mathbb{C}[s]; t(\lambda) \neq 0 \} \subset \mathbb{C}(s),
\]

\[
A/m_\lambda \cong A/m_\lambda/(m_\lambda)_{m_\lambda}.
\]

Recall that a \( \mathbb{C}[s] \)-module \( \mathcal{F} \) is \textit{injective} if the contravariant duality functor

\[
D := \text{Hom}_{\mathbb{C}[s]}(-, \mathcal{F}), \quad \left( M' \xrightarrow{f} M \right) \mapsto \left( D(M) = \text{Hom}_{\mathbb{C}[s]}(M, \mathcal{F}) \xrightarrow{D(f)} D(M'), \varphi \mapsto \varphi f \right)
\]

is exact on the category of \( \mathbb{C}[s] \)-modules or transforms monomorphisms into epimorphisms.

\textbf{Lemma and Definition 3.3} An injective \( A \)-module \( \mathcal{F} \) is a cogenerator if it satisfies the following equivalent conditions:

1. If \( M \) is a non-zero module then so is \( \text{Hom}_{\mathbb{C}[s]}(M, \mathcal{F}) \).

2. The module \( \mathcal{F} \) contains all simple modules \( \mathbb{C}[s]/m, \ m \in \text{Max}(\mathbb{C}[s]), \) up to isomorphism.

3. A sequence of \( A \)-modules

\[
M' \xrightarrow{f} M \xrightarrow{g} M''
\]

is exact if and only if the dual sequence

\[
D(M'' \xrightarrow{D(g)} D(M) \xrightarrow{D(f)} D(M'))
\]

is exact.

\textbf{Theorem 3.4} [27, Theorem 2.54] [28, Theorem 6.6] The three \( \mathbb{C}[s] \)-modules

\[
\mathcal{F}_{\text{cont}} = \oplus_{\lambda \in \mathbb{C}^r} \mathbb{C}[t] e^{\lambda t} \subset C^\infty(\mathbb{R}^r) \subset D'(\mathbb{R}^r)
\]

are injective cogenerators.
The difficult injectivity of the two larger modules was established by L. Ehrenpreis [12], B. Malgrange [25] and V. Palamodov [30] in the beginning 1960s and was called the fundamental principle for these (and other) function modules. The result implies the divisibility of these modules and therefore the important existence of a fundamental solution $y \in D'$ of $f \circ y = \delta$ for any non-zero polynomial or differential operator $f$. The injectivity of the module of polynomial-exponential functions can be shown more easily by reduction to the discrete case as explained below.

It is easy to see that $\mathbb{C} e^{\lambda \cdot s}$ and $\mathbb{C}[t] e^{\lambda \cdot s}$ are $\mathbb{C}[s]$-submodules of $\mathcal{D}'$, $$A/m_\lambda \cong \mathbb{C} e^{\lambda \cdot s}$$

is a $\mathbb{C}[s]$-isomorphism and the inclusion $\mathbb{C} e^{\lambda \cdot s} \subset \mathbb{C}[t] e^{\lambda \cdot s}$ is essential, i.e.,

$$\forall \text{ non-zero } y \in \mathbb{C}[t] e^{\lambda \cdot s} \exists f \in \mathbb{C}[s] \text{ such that } 0 \neq f \circ y \in \mathbb{C} e^{\lambda \cdot s}.$$ Hence for all polynomials $f \in \mathbb{C}[s] \setminus m_\lambda$ the maps

$$f \circ : \mathbb{C} e^{\lambda \cdot s} \cong \mathbb{C} e^{\lambda \cdot t} \text{ and } f \circ : \mathbb{C}[t] e^{\lambda \cdot s} \cong \mathbb{C}[t] e^{\lambda \cdot t}$$

are isomorphisms and therefore $\mathbb{C}[t] e^{\lambda \cdot s}$ is an $A_{m_\lambda}$-module.

**Corollary 3.5** [28] For each $\lambda \in \mathbb{C}^r$ the module $\mathbb{C}[t] e^{\lambda \cdot s}$, $\lambda \in \mathbb{C}^r$, is injective and indecomposable as $A$- and as $A_{m_\lambda}$-module. More precisely, the modules $\mathcal{F}_{\text{cont}} = \bigoplus_{\lambda \in \mathbb{C}^r} \mathbb{C}[t] e^{\lambda \cdot s}$ of polynomial-exponential functions resp. $\mathbb{C}[t] e^{\lambda \cdot s}$ are the minimal injective cogenerators over $A$ resp. over $A_{m_\lambda}$, up to isomorphism. This property will be used frequently later on.

The following lines contain the analogous definitions for the discrete case of partial difference equations.

**Lemma and Definition 3.6 (Discrete multidimensional signals)** [27, sections 4.12, 4.13] [28, Theorem 1.14] As signal space in the discrete case we use the space

$$\mathbb{C}^N = \{ y = (y_\mu)_{\mu \in N^r} : N^r \to \mathbb{C}, \mu = (\mu_1, \ldots, \mu_r) \mapsto y(\mu) := y_\mu \}$$

of multisequences or functions $y$ in $r$ independent discrete variables $\mu_\rho \in \mathbb{N}$ with the $\mathbb{C}[s]$-action

$$(s^\mu \circ y)(\nu) := y(\mu + \nu), \quad s^\mu := s_{\mu_1}^{\mu_1} \ast \ldots \ast s_{\mu_r}^{\mu_r},$$

by left shifts. The $\mathbb{C}[s]$-isomorphism

$$\text{Hom}_{\mathbb{C}[s]}(\mathbb{C}[s], \mathbb{C}) \cong \mathbb{C}^N, \quad \varphi \mapsto y, \quad y(\mu) := \varphi(s^\mu),$$

easily implies that also $\mathbb{C}^N$ is an injective $\mathbb{C}[s]$-cogenerator, and so is the submodule

$$\mathcal{F}_{\text{dis}} := \mathbb{C}^N_{\text{dis}} := \left\{ y \in \mathbb{C}^N : \dim_{\mathbb{C}}(\mathbb{C}[s] \circ y) < \infty \right\}$$

which is again the space of polynomial-exponential sequences.
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This definition can be extended to sequences \( y = (y_\mu)_{\mu \in \mathbb{Z}^r} \in \mathbb{C}^{\mathbb{Z}^r} \) [27, 29, 11, 47]. However, the stability theory of the present paper has not yet been developed for this case.

The discrete polynomial-exponential sequences have the following form.

**Lemma and Definition 3.7** [28, Theorem 1.25] For \( \alpha \in \mathbb{C} \), \( k, i \in \mathbb{N} \), \( \lambda \in \mathbb{C}^r \), \( \mu, \nu \in \mathbb{N}^r \) define \( e_{\alpha,k} \in \mathbb{C}^\mathbb{N} \) and \( e_{\lambda,\mu} \in \mathbb{C}^{\mathbb{N}^r} \) by

\[
e_{\alpha,k}(i) := \begin{cases} \delta_{i,k} & \text{if } \alpha = 0 \\ \frac{1}{\alpha} i^{k-1} & \text{if } \alpha \neq 0 \end{cases} \quad \text{and} \quad e_{\lambda,\mu}(\nu) := \prod_{\rho=1}^{r} e_{\lambda\rho,\mu\rho}(\nu_\rho).
\]

Then

\[
(s - \lambda)^\nu \circ e_{\lambda,\mu} = \begin{cases} e_{\lambda,\mu-\nu} & \text{if } \mu \in \nu + \mathbb{N}^r \\ 0 & \text{otherwise}
\end{cases}
\]

and

\[
\mathcal{F}_{\text{dis}} := \mathbb{C}^{\mathbb{N}^r}_{\text{if}} := \left\{ y \in \mathbb{C}^{\mathbb{N}^r} ; \dim_{\mathbb{C}}(\mathbb{C}[s] \circ y) < \infty \right\} = \bigoplus_{\lambda \in \mathbb{C}^r, \mu \in \mathbb{N}^r} \mathbb{C} e_{\lambda,\mu}.
\]

In the one-dimensional case, i.e., \( r = 1 \), these are the sequences that are polynomial-exponential after finitely many time steps. For \( r \in \mathbb{N} \) arbitrary, there exists \( \mu \in \mathbb{N}^r \) such that a sequence in \( \mathcal{F}_{\text{dis}} \) is polynomial-exponential on \( \mu + \mathbb{N}^r \). With \( t^\mu := t_1^{\mu_1} \ast \ldots \ast t_r^{\mu_r} \) the \( \mathbb{C}[s] \)-linear isomorphism

\[
\mathbb{C}^{\mathbb{N}^r} \cong \mathbb{C}[t], \quad y = (y_\mu)_{\mu \in \mathbb{N}^r} \mapsto \sum_{\mu \in \mathbb{N}^r} y_\mu \frac{t^\mu}{\mu!},
\]

induces the isomorphisms

\[
\mathcal{F}_{\text{dis}} = \bigoplus_{\lambda,\mu} \mathbb{C} e_{\lambda,\mu} \cong \mathcal{F}_{\text{cont}} = \bigoplus_{\lambda} \mathbb{C}[t] e^{\lambda t} \text{ and }
\]

\[
\bigoplus_{\mu \in \mathbb{N}^r} \mathbb{C} e_{\lambda,\mu} \cong \mathbb{C}[t] e^{\lambda t}, \quad e_{\lambda,\mu} \mapsto \frac{t^\mu}{\mu!} e^{\lambda t}.
\]

The isomorphism \( \mathcal{F}_{\text{dis}} \cong \mathcal{F}_{\text{cont}} \) shows that the minimal injective cogenerator property of \( \mathcal{F}_{\text{cont}} \) follows from that of \( \mathcal{F}_{\text{dis}} \).

### 4 Stable one-dimensional systems

In this section we recall the stability theory for the simplest one-dimensional systems as a model for the following multidimensional theory and start with the continuous case.

Let

\[
r := 1, \quad s := s_1, \quad t := t_1 \in \mathbb{R}, \quad \lambda := \lambda_1 \in \mathbb{C},
\]

\[
A \in \mathbb{C}^{p \times p}, \quad \text{Spec}(A) := \{\text{eigenvalues of }A\},
\]

stable region \( \Lambda_1 := \{\lambda \in \mathbb{C} \mid \Re(\lambda) < 0\} \).
We consider the simple differential system
\[ \dot{y} = Ay + u, \quad u, y \in D^p, \]
with the solutions
\[ B^0 := \left\{ y \in D'(\mathbb{R})^p; \quad \dot{y} = Ay \right\} \subset \bigoplus_{\lambda \in \text{Spec}(A)} \mathbb{C}[t] e^{\lambda t}, \]
\[ y(t) = e^{tA} y(0) + \int_0^t e^{(t-\tau)A} u(\tau) d\tau, \quad \text{if } u \text{ is continuous.} \]

The following theorem is the standard result on the stability of these systems.

**Theorem 4.1**

1. The following assertions are equivalent:
   
   (i) Analysis: 
   \[ B^0 := \left\{ y \in D'(\mathbb{R})^p; \quad \dot{y} = Ay \right\} \subset \bigoplus_{\lambda \in \Lambda_1} \mathbb{C}[t] e^{\lambda t}, \]
   
   (ii) \( B^0 \) is asymptotically stable, i.e., for all \( y \in B^0 \): \( \lim_{t \to \infty} y(t) = 0 \).

   (iii) Geometry: 
   \[ \text{Spec}(A) \subset \Lambda_1 = \{ \lambda \in \mathbb{C}; \quad \Re(\lambda) < 0 \}. \]

2. If the conditions of the first item are satisfied the system is BIBO stable (bounded input/bounded output stable), i.e., if the differential equation
   \[ \dot{y} = Ay + u \]
   is satisfied and if \( u \) is continuous and bounded for \( t \geq 0 \) then so is \( y \).

We reformulate the preceding data in order to get a clue for a reasonable multidimensional generalization:

\[ P := s \text{id}_p - A, \quad Q := \text{id}_p \in \mathbb{C}[s]^{p \times p}, \quad \text{rank}(s \text{id}_p - A) = p, \]
\[ \dot{y} = Ay + u \iff P \circ y = P \left( \frac{d}{dt} \right) y = Q \circ u, \quad \text{(4.1)} \]
\[ \text{Spec}(A) = \{ \lambda \in \mathbb{C}; \quad \text{rank}(P(\lambda)) < p \}. \]

The rank of a polynomial matrix is that as matrix with coefficients in the field \( \mathbb{C}(s) \) of rational functions and therefore the maximal number of linearly independent columns. In the discrete case we consider the stable region \( \Lambda_1 := \{ \lambda \in \mathbb{C}; \quad |\lambda| < 1 \} \) and the difference system

\[ y(i+1) = Ay(i) + u(i), \quad u, y \in \mathbb{C}^N, \]

with the solutions
\[ B^0 := \left\{ y \in (\mathbb{C}^p)^N; \quad y(i+1) = Ay(i) \right\} \subset \bigoplus_{\lambda \in \text{Spec}(A), \ i \in \mathbb{N}} \mathbb{C}^p e_{\lambda,i} \subset F^p_{\text{dis}}, \]
\[ y(i) = A^i y(0) + \sum_{j,k,j+k=i-1} A^j u(k). \]

Theorem 4.1 remains valid with the new data
\[ B^0 \subset \bigoplus_{\lambda \in \Lambda_1, \ i \in \mathbb{N}} \mathbb{C}^p e_{\lambda,i} \quad \text{and} \quad \text{Spec}(A) \subset \{ \lambda \in \mathbb{C}; \quad |\lambda| < 1 \}. \]
5 Multidimensional IO behaviors

In this section we specify the multidimensional IO systems which we study in this paper, in particular in context with the feedback stabilization of section 2. We generalize the data of equation (4.1). The behaviors have an algebraic, an analytic and a geometric aspect, and we discuss them all.

**Algebra:** Consider a polynomial matrix

\[ R \in \mathbb{C}[s]^{k \times l} \text{ of } \text{rank}(R) =: p \quad \text{and} \quad m := l - p. \]

Various choices of \( p \) columns of \( R \) are linearly independent and are called an input/output or IO structure in systems theory. This terminology will be justified below. We make such a choice. After a possible permutation of the columns the first \( p \) columns of \( R \) are the IO structure and the matrix is written as

\[ R = (P, -Q) \in \mathbb{C}[s]^{k \times (p+m)} \text{ with } \text{rank}(P) = \text{rank}(R) = p. \]

Hence there is a unique rational matrix

\[ H \in \mathbb{C}(s)^{p \times m} \text{ with } PH = Q \]

which is called the transfer matrix in systems theory.

The matrices \( R = (P, -Q) \) and \( P \) give rise to their row modules and corresponding factor modules

\[ U := \mathbb{C}[s]^{1 \times k}R = \sum_{i=1}^{k} \mathbb{C}[s]R_i \subset \mathbb{C}[s]^{1 \times (p+m)} \quad \text{and} \]

\[ U^0 := \mathbb{C}[s]^{1 \times k}P = \sum_{i=1}^{k} \mathbb{C}[s]P_i \subset \mathbb{C}[s]^{1 \times p}, \quad \text{as well as} \]

\[ M := \mathbb{C}[s]^{1 \times (p+m)}/U, \quad \text{and} \quad M^0 := \mathbb{C}[s]^{1 \times p}/U^0. \]

The isomorphism

\[ \mathbb{C}(s) \otimes_{\mathbb{C}[s]} M \cong \mathbb{C}(s)^{1 \times (p+m)}/\mathbb{C}(s)^{1 \times k}R \]

motivates to define the factor module’s rank as

\[ \text{rank}(M) := \dim_{\mathbb{C}(s)}(\mathbb{C}(s) \otimes_{\mathbb{C}[s]} M) = m. \]

Since \( M^0 \) is a torsion module we have

\[ \mathbb{C}(s) \otimes_{\mathbb{C}[s]} M^0 = 0 \quad \text{and} \quad \text{rank}(M^0) = 0. \]

The last row shows that \( M^0 \) is a torsion module or has a non-zero annihilator ideal

\[ \text{ann}_{\mathbb{C}[s]}(M^0) := \{ f \in \mathbb{C}[s]; \ f M^0 = 0 \text{ or } f \mathbb{C}[s]^{1 \times p} \subseteq U^0 \}. \quad (5.1) \]
Geometry [27, p. 155ff]: We use the algebraic data, i.e., matrices and modules, from above. For every point $\lambda \in \mathbb{C}^r$ the module $M$ and the maximal ideal $m_\lambda$ give rise to the quotient module

$$M_{m_\lambda} = \left\{ \frac{x}{t} : x \in M, t \in \mathbb{C}[s], t(\lambda) \neq 0 \right\}$$

over the local ring $\mathbb{C}[s]_{m_\lambda}$.

For any ideal $a \subseteq \mathbb{C}[s]$ we consider its variety

$$V(a) := \{ \lambda \in \mathbb{C}^r ; \forall f \in a : f(\lambda) = 0 \} \subseteq \mathbb{C}^r.$$

We obtain two characteristic varieties which are essentially connected with $M$ and $M^0$, viz.

- (the characteristic variety of $M$) := char($M$)
  $$= \{ \lambda \in \mathbb{C}^r ; \text{rank}(R(\lambda)) < p = \text{rank}(R) \}$$
  $$= \{ \lambda \in \mathbb{C}^r ; M_{m_\lambda} \text{ is not free} \}$$

- (the characteristic variety of $M^0$) := char($M^0$)
  $$= \{ \lambda \in \mathbb{C}^r ; \text{rank}(P(\lambda)) < p = \text{rank}(P) \}$$
  $$= \{ \lambda \in \mathbb{C}^r ; M^0_{m_\lambda} \neq 0 \}$$
  $$= V(\text{ann}_{\mathbb{C}[s]}(M^0))$$

with char$(M) \subset$ char$(M^0)$.

In systems theory these varieties are also called the varieties of rank singularities for obvious reasons. In the one-dimensional example $P := \text{id}_p - A, A \in \mathbb{C}^{p \times p}$, the characteristic variety char$(M^0)$ coincides with the spectrum Spec$(A)$. Therefore one expects that the characteristic varieties determine the stability properties of a multidimensional behavior, and this is indeed the case.

Analysis: The data from above are in force. Let $\mathcal{F}$ denote one of the injective $\mathbb{C}[s]$-cogenerators discussed in section 3 in the continuous or discrete case. The multidimensional $\mathcal{F}$-behavior defined by $R$ is the solution module of the linear system $R \circ w = 0$ of partial differential or difference equations with constant coefficients, more precisely

$$\mathcal{B} := \{ w = (w_1, \ldots, w_l)^t \in \mathcal{F}^l ; R \circ w = 0 \}$$

$$= \ker \left( R_0 : \mathcal{F}^l \to \mathcal{F}^k, w \mapsto R \circ w \right) \subset \mathcal{F}^l.$$

For obvious reasons J. C. Willems calls the preceding form a kernel representation of $\mathcal{B}$. With the standard basis

$$\delta_i := (0, \ldots, 0, 1, 0, \ldots, 0) \in \mathbb{C}[s]^{1 \times l}, \quad i = 1, \ldots, l,$$

one obtains the $\mathbb{C}[s]$-isomorphism, due to B. Malgrange [25],

$$\text{sol}_\mathcal{F}(M) := \text{Hom}_{\mathbb{C}[s]}(M, \mathcal{F}) \cong \mathcal{B}, \varphi \mapsto w, \varphi(\delta_i) = w_i.$$
The significance of this simple isomorphism lies in the fact that it connects algebra with analysis and thus was one origin of Algebraic Analysis. In the same fashion one obtains the behavior

\[ B^0 := \left\{ y = (y_1, \ldots, y_p)^\top \in \mathcal{F}^p; \ P \circ y = 0 \right\} \text{ and } \]

\[ \text{sol}_\mathcal{F}(M^0) := \text{Hom}_{\mathbb{C}[s]}(M^0, \mathcal{F}) \cong B^0. \]

The IO structure and the associated decompositions \( R = (P, -Q) \) and \( w := \left( \begin{array}{c} y \\ u \end{array} \right) \in \mathcal{F}^1 = \mathcal{F}^{p+m} \) of the matrix \( R \) resp. the function vectors yield the alternative IO representation of \( B \):

\[ B := \left\{ \left( \begin{array}{c} y \\ u \end{array} \right) \in \mathcal{F}^{p+m}; \ R \circ \left( \begin{array}{c} y \\ u \end{array} \right) = 0 \text{ or } P \circ y = Q \circ u \right\}, \]

\[ B^0 \cong B \cap (\mathcal{F}^p \times \{0\}), \ y \mapsto \left( \begin{array}{c} y \\ 0 \end{array} \right). \]

We interpret \( P \circ y = Q \circ u \) as an inhomogeneous linear system with right side \( Q \circ u \) and \( P \circ y = 0 \) as the associated homogeneous system with solution module \( B^0 \).

**Theorem and Definition 5.1** [27, Theorem 2.69 and Corollary 2.27]

1. For each \( u \in \mathcal{F}^m \) there is a \( y \in \mathcal{F}^p \) such that

\[ P \circ y = Q \circ u \text{ or } \left( \begin{array}{c} y \\ u \end{array} \right) \in B, \]

i.e., one may select the component \( u \) of a trajectory in \( B \) freely or, in other terms, the sequence

\[
\begin{array}{ccccccc}
0 & \rightarrow & B^0 & \xrightarrow{\text{inj}} & B & \xrightarrow{\text{proj}} & \mathcal{F}^m & \rightarrow & 0 \\
\downarrow & & y & \mapsto & \left( \begin{array}{c} y \\ u \end{array} \right) & \rightarrow & u
\end{array}
\]

is exact. It is this property which justifies to call \( u \) the input and \( y \) a corresponding output of \( B \) for the chosen IO structure. The components \( u_i \) of the input are also called free.

2. The module \( U \) and therefore also the module \( M \) are uniquely determined by the solution module \( B \), indeed

\[ U = \mathbb{C}[s]^{1 \times k} R = B^\perp := \left\{ x \in \mathbb{C}[s]^{1 \times (p+m)}; \ \forall w \in B: \ x \circ w = 0 \right\}. \]

J. F. Pommaret calls the elements of \( M \) the observables of \( B \). We also define the characteristic varieties

\[ \text{char}(B) := \text{char}(M) \quad \text{and} \quad \text{char}(B^0) := \text{char}(M^0). \]

3. The equality \( \text{ann}_{\mathbb{C}[s]}(B^0) = \text{ann}_{\mathbb{C}[s]}(M^0) \neq 0 \) shows that there is a non-zero polynomial \( f \) such that \( f \circ y = 0 \) for all \( y \in B^0 \), i.e., all solutions of the homogeneous
system $P \circ y = 0$ satisfy the same differential resp. difference equation. The behavior $B^0$ is called the autonomous part of $B$ for the chosen IO structure. The behavior $B$ is called autonomous if

$$B = B^0 \iff M = M^0 \iff M \text{ is a torsion module.}$$

All statements of the preceding theorem follow from the injective cogenerator property of $\mathcal{F}$.

### 6 Stable multidimensional systems

In this section we are going to define and characterize stable multidimensional systems in the continuous and discrete cases. The data and notations are those of the preceding section.

We need an analogue of the stable region $\{\lambda \in \mathbb{C}; \Re(\lambda) < 0\}$ of the one-dimensional situation of section 4. For maximal flexibility we define stability for an arbitrary disjoint decomposition

$$C^r = \Lambda_1 \sqcup \Lambda_2, \quad \Lambda_1 =: \text{stable region}, \quad \Lambda_2 =: \text{unstable region.} \quad (6.1)$$

The standard example is

$$\Lambda_2 := \begin{cases} \{\lambda \in \mathbb{C}^r; \forall \rho = 1, \ldots, r : \Re(\lambda_\rho) \geq 0\} & \text{in the continuous case.} \\ \{\lambda \in \mathbb{C}^r; \forall \rho = 1, \ldots, r : |\lambda_\rho| \geq 1\} & \text{in the discrete case.} \end{cases} \quad (6.2)$$

We define the saturated multiplicatively closed set

$$T := \{t \in \mathbb{C}[s]; V(t) \subset \Lambda_1\} = \{t \in \mathbb{C}[s]; \forall \lambda \in \Lambda_2 : t(\lambda) \neq 0\} = \bigcap_{\lambda \in \Lambda_2} (\mathbb{C}[s] \setminus m_\lambda) = \{\text{stable polynomials}\}$$

of stable polynomials with respect to the decomposition (6.1) and the ensuing quotient ring

$$\mathbb{C}[s]_T := \left\{\frac{a}{t}; \ a \in \mathbb{C}[s], \ t \in T\right\} = \bigcap_{\lambda \in \Lambda_2} \mathbb{C}[s]_{m_\lambda} = \{\text{stable rational functions}\} \subseteq \mathbb{C}(s)$$

of stable rational functions.

The next lemma shows that the stable region $\Lambda_1 := \{z \in \mathbb{C}; \Re(z) < 0\}^r$ is no suitable generalization of the one-dimensional situation and uninteresting.
Lemma 6.1 For \( r > 1 \) consider disjoint decompositions
\[
C = \Lambda_1, \rho \uplus \Lambda_2, \rho, \rho = 1, \ldots, r, \quad \text{with} \quad \text{infinite} \ \Lambda_2, \rho.
\]
\[
\Lambda_1 := \prod_{\rho=1}^{r} \Lambda_1, \rho, \quad C^r = \Lambda_1 \uplus \Lambda_2.
\]
Then the non-zero constants are the only stable polynomials with respect to this decomposition, i.e.,
\[
T := \{ t \in \mathbb{C}[s]; \ V(t) \subseteq \Lambda_1 \} = \mathbb{C} \setminus \{0\}.
\]
Proof. Define \( s' = (s_1, \ldots, s_{r-1}) \), hence \( s = (s', s_r) \), and let \( f = f(s) = f(s', s_r) \) be a stable polynomial, i.e., \( V(f) \subseteq \Lambda_1 := \prod_{\rho=1}^{r} \Lambda_1, \rho \) and \( f \neq 0 \). The assumption implies that for every \( \lambda_2, r \in \Lambda_2, r \) the polynomial
\[
f(s', \lambda_2, r) \text{ has no zero, hence is constant, i.e., } f(s', \lambda_2, r) = f(0, \lambda_2, r), \quad \text{and} \quad C^{r-1} \times \Lambda_2, r \subseteq V(f - f(0, s_r)), \quad \text{hence } f - f(0, s_r) = 0
\]
since this polynomial is zero on a product of infinite subsets of \( \mathbb{C} \). Thus
\[
f = g(s_r), \quad 0 \neq g \in \mathbb{C}[s_r], \quad \text{hence } V(f) = C^{r-1} \times V(g).
\]
Again by assumption this set is empty and hence \( f = g(s_r) \) is a non-zero constant. \( \square \)

For the next theorem we use the injective cogenerator \( F \) of polynomial-exponential functions in the continuous resp. discrete cases from (3.1) and Lemma 3.7, i.e.,
\[
F := \begin{cases}
F_{\text{cont}} = \oplus_{\lambda \in \mathbb{C}^{\mathbb{C}}} \mathbb{C}[t] e_{\lambda t} & \text{in the continuous case} \\
F_{\text{dis}} = \oplus_{\lambda \in \mathbb{C}^{\mathbb{C}}} \oplus_{\mu \in \mathbb{N}} \mathbb{C} e_{\lambda, \mu} & \text{in the discrete case}
\end{cases}
\]
and the submodules
\[
F_i := \begin{cases}
\oplus_{\lambda \in \mathbb{C}^{\mathbb{C}}} \mathbb{C}[t] e_{\lambda t} & \text{in the continuous case, } i = 1, 2, \text{ hence} \\
\oplus_{\lambda \in \mathbb{C}^{\mathbb{C}}} \oplus_{\mu \in \mathbb{N}} \mathbb{C} e_{\lambda, \mu} & \text{in the discrete case, } i = 1, 2, \text{ hence}
\end{cases}
\]
\[
F = F_1 \oplus F_2, \quad B := B_1 \oplus B_2 \text{ with } B_i := B \cap F_i^{p+m}.
\]

With these data we obtain the analogue of Theorem 4.1

Theorem and Definition 6.2 (Stable multidimensional behaviors) [26, Theorems 5.1, 5.3, 5.4] The data are those from section 5 and from above.

1. The following statements are equivalent:
   (i) Analysis: \( B^0 \subset F_i^p \).
   (ii) Geometry: \( \text{char}(B^0) \subset \Lambda_1 \) = stable region.
   (iii) Algebra:
      (a) The transfer matrix is stable, i.e., \( H \in \mathbb{C}[s]^{p \times m}_T \), and
      (b) for each \( \lambda \in \Lambda_2 \) the \( \mathbb{C}[s]_{m_\lambda} \)-module \( M_{m_\lambda} \) is torsionfree.
Under these conditions $B$ is called stable with respect to the decomposition (6.1).

2. The differential resp. difference operators $f : F_2 \to F_2$, $f \in T$, are isomorphisms, i.e., $F_2$ is a $C[s]_T$-module. For a stable system $B$ as in 1. this implies the isomorphism

$$F_2^m \cong B_2 = \text{graph}(H \circ |F_2^m|)$, \quad \text{hence} \quad B = B_1 \oplus \text{graph}(H \circ |F_2^m|) \ni \begin{pmatrix} y \\ u \end{pmatrix} = \begin{pmatrix} y_1 \\ u_1 \end{pmatrix} + \begin{pmatrix} H \circ u_2 \\ u_2 \end{pmatrix}.$$

In the usual suggestive language the first resp. the second summand of the trajectory $w \in B$ are called its stable part resp. its steady state.

3. If $B$ is stable, $d \in \mathbb{N}$, $\Gamma \subset \Lambda_2$ is finite and $u \in F_2^m$ has the form

$$u = \sum_{\gamma \in \Gamma} u_\gamma(t) e^{\gamma t} \quad \text{with} \quad u_\gamma(t) \in C[t]^m \quad \text{and} \quad \deg_t(u_\gamma) \leq d$$

where $\deg_t$ denotes the total degree in $C[t]$ then the output $H \circ u$ has the same form. This is a certain equivalent for BIBO stability. An analogous result holds in the discrete case.

4. If in the continuous case $i \mathbb{R}^r \subseteq \Lambda_2$ and if $G$ denotes one of the spaces

$$G := \begin{cases} S & \text{of rapidly decreasing } C^\infty \text{-functions} \\ O & \text{of slowly increasing } C^\infty \text{-functions} \\ S' & \text{of temperate distributions} \\ O' & \text{of rapidly decreasing distributions} \end{cases}$$

then $G$ is also a $C[s]_T$-module. The stability of $B := B_{\mathcal{F}_{\text{cont}}}$ then implies the $C[s]_T$-monomorphism

$$G^m \to B_G := \left\{ \begin{pmatrix} y \\ u \end{pmatrix} \in G^{p+m}; \quad P \circ y = Q \circ u \right\}, \quad \text{hence} \quad \begin{pmatrix} H \circ u \\ u \end{pmatrix}, \quad \text{(6.3)}$$

and this is another variant of BIBO stability in the multidimensional situation. The map (6.3) is even an isomorphism if the set $\Lambda_2$ is ideal-convex (see section 7).

It is not clear (to us) how the condition 1.(i) of the preceding theorem can be generalized to other function spaces $G$ like those of $C^\infty$-functions or distributions instead of polynomial-exponential functions. In this context the so-called integral representation theorem of L. Ehrenpreis and V. Palamodov [3, 12, 30] could be useful. This theorem expresses solutions in $B_2^q$ as integrals over polynomial-exponential solutions with respect to measures which are supported in the characteristic variety. The difficulties in this approach are of an analytic nature. Due to these difficulties we have applied the space of polynomial-exponential functions for the formulation of Theorem 6.2.
7 Gabriel localization

P. Gabriel developed his categorical localization theory for the construction of non-commutative quotient rings 45 years ago. The theory was exposed by B. Stenström in [44] and is also explained in [26, §3]. There we observed that the theory is also non-trivial for the Noetherian polynomial algebra $A = \mathbb{C}[s]$ and that it is significant for the stabilization of IO systems. Here we give a short introduction into this special application.

The assumptions and notations of sections 5 and 6 are in force. It is a standard and simple observation that the quotient module $M_{T}$ of an $A$-module $M$ can be constructed as a colimit or direct limit over the ordered set $\{At; t \in T\}$ of ideals which is directed downwards via

$$At_1 \subseteq At_1 \cap At_2.$$

The construction is

$$M_T \cong \text{colim}_{At; t \in T} \text{Hom}_A(At, M) \quad \text{with} \quad \text{Hom}_A(At, M) \cong M, \quad \alpha \mapsto \alpha(t).$$

Recall that the functor $M \mapsto M_T$ is exact. In the situation of the present paper we replace the set $\{At; t \in T\}$ by the downward directed set

$$T := \{a \subseteq A; \forall \lambda \in \Lambda_2 : (A/a)_{m_\lambda} = 0\} = \{a \subseteq A; \forall \lambda \in \Lambda_2 : a \cap (A \setminus m_\lambda) \neq \emptyset\} = \{a \subseteq A; V(a) \cap \Lambda_2 = \emptyset\},$$

hence $At \in T$ for all $t \in T = \bigcap_{\lambda \in \Lambda_2} (A \setminus m_\lambda)$.

The set $T$ is the basis of neighborhoods of 0 of a unique linear topology on $A$ which is called a Gabriel topology. It induces the new left-exact Gabriel localization functor on the category of $A$-modules

$$M \mapsto M_\Sigma = \text{colim}_{a \in \Sigma} \text{Hom}_A(a, M).$$

The standard colimit construction of $M_\Sigma$ is

$$M_\Sigma = \bigcup_{a \in \Sigma} \text{Hom}_A(a, M)/ \sim \ni [\alpha : a \to M] \quad \text{where}$$

$$[\alpha_1 : a_1 \to M] \sim [\alpha_2 : a_2 \to M] :\iff$$

$$\exists b \in \Sigma \text{ such that } b \subseteq a_1 \cap a_2 \text{ and } \alpha_1|b = \alpha_2|b.$$
in particular \( A_T \cong A_\Sigma \) or \( A_T = A_\Sigma \) by identification, which induce canonical isomorphisms

\[
(M_T)_\Sigma \cong M_\Sigma, \quad M_m^\lambda \cong (M_T)^m_\lambda, \quad \mathbb{C}(s) \otimes_A M \cong \mathbb{C}(s) \otimes_A M_T \cong \mathbb{C}(s) \otimes_A M_\Sigma.
\]

For any finitely generated module \( M = A^{1 \times l}/U \) and the injective modules

\[
F_\lambda := \mathbb{C}[t]e^{\lambda \ast t} \cong \bigoplus_{\mu \in \mathbb{N}} \mathbb{C}e_{\lambda, \mu}, \quad F_2 = \bigoplus_{\lambda \in A_2} F_\lambda,
\]

there are canonical isomorphisms

\[
B_2 := B \cap F_2^l \cong \text{Hom}_A(M, F_2) \cong \text{Hom}_{A_T}(M_T, F_2) \cong \bigoplus_{\lambda \in A_2} \text{Hom}_{A_m^\lambda}(M_m^\lambda, F_\lambda).
\]

The decisive advantage of Gabriel’s localization \( M_\Sigma \) over the usual quotient module \( M_T \) is that the canonical map

\[
M_\Sigma \xrightarrow{\text{can}} \prod_{\lambda \in A_2} M_m^\lambda \text{ is injective, and hence}
\]

\[
M_\Sigma = 0 \iff \forall \lambda \in A_2 : M_m^\lambda = 0 \iff B_2 = B \cap F_2^l = 0.
\]

(7.1)

The left exactness of \((-)_T\) on \( A\)-modules, the monomorphism (7.1) and \( A_T = A_\Sigma \) also imply the representation

\[
U_\Sigma = A_T^{1 \times l} \bigcap \bigcap_{\lambda \in A_2} U_m^\lambda \subseteq A_T^{1 \times l} \subseteq (s)^{1 \times l}
\]

for any submodule \( U \subseteq A^{1 \times l} \) and then \( U_\Sigma = U_{\text{st}, T} \) where

\[
U_{\text{st}} := A^{1 \times l} \bigcap \bigcap_{\lambda \in A_2} U_m^\lambda = A^{1 \times k_{\text{st}}} R_{\text{st}}, \quad R_{\text{st}} \in A^{k_{\text{st}} \times (p+m)}.
\]

or, in other terms, the rows of \( R_{\text{st}} \) are \( A \)-generators of \( U_{\text{st}} \) and \( A_T \)-generators of \( U_\Sigma \).

In general, it is difficult to compute these, but in important special cases they can be obtained in the following fashion. The data are again those of sections 5 and 6, i.e.,

\[
U = A^{1 \times k}(P, -Q) \subseteq A^{1 \times (p+m)}, \quad M = A^{1 \times (p+m)}/U, \quad PH = Q.
\]

Define

\[
U_{\text{cont}} := \{(\xi, \eta) \in A^{1 \times (p+m)}; \xi H + \eta = 0\} = A^{1 \times k_{\text{cont}}} R_{\text{cont}}.
\]

Then, \( t(M) := U_{\text{cont}}/U \subseteq M \) is the torsion submodule of \( M \). Let

\[
a := \text{ann}_A(t(M))
\]
be its annihilator ideal. Both $U_{\text{cont}}$ and $a$ can be computed via Gröbner bases. The IO behavior

\[
B_{\text{cont}} := \left\{ \begin{array}{l}
y \\
u
\end{array} \in \mathcal{F}^{p+m}; \ P_{\text{cont}} \circ y = Q_{\text{cont}} \circ u \right\}
\]

with

\[
(P_{\text{cont}}, -Q_{\text{cont}}) := R_{\text{cont}} \in A^{k_{\text{cont}} \times (p+m)}
\]

is the least or the unique controllable IO behavior with transfer matrix $H$.

**Theorem 7.1** [26, Theorem, 5.14] Data as just introduced.

1. The following properties are equivalent:

   (i) All localized modules $M_{m_\lambda}, \lambda \in \Lambda_2$, are torsionfree.

   (ii) $V(a) \cap \Lambda_2 = \emptyset$ or, equivalently, $a \in \mathfrak{T}$.

   If these equivalent properties hold then

   \[
   U_{\mathfrak{T}} = U_{\text{cont},T}, \quad \text{hence} \quad U_{\text{st}} = U_{\text{cont}} \quad \text{and} \quad R_{\text{st}} := R_{\text{cont}}.
   \]

2. The module $M_T$ is torsionfree if and only if $a \cap T \neq \emptyset$. If this is the case then

   \[
   U_{\mathfrak{T}} = U_{\text{cont},T} = U_T \quad \text{and} \quad M_T \cong M := A_T^{1 \times p}H + A_T^{1 \times m} \subset \mathbb{C}(s)^{1 \times m}. \quad (7.2)
   \]

The module $M$ is an $A_T$-lattice in $\mathbb{C}(s)^{1+m}$.

For the following notion of ideal convexity we make two simple observations.

1. For any ideal $a \subseteq A = \mathbb{C}[s]$ we get

   \[
a \cap T \neq \emptyset \Rightarrow V(a) \cap \Lambda_2 = \emptyset. \quad (7.3)
   \]

2. If $m$ is a maximal ideal of $A$ with $m \cap T = \emptyset$ then $m_T$ is a maximal ideal of $A_T$ and therefore

   \[
   \{(m_\lambda)_T; \lambda \in \Lambda_2\} \subseteq \text{Max} \left( \mathbb{C}[s]_T \right).
   \]

**Theorem and Definition 7.2 (Perfect localization and ideal-convexity)** [44, p. 231] [43, p. 25] [26, Theorem 5.6] [51]

1. Under the preceding assumptions the following properties are equivalent for the polynomial ring $A = \mathbb{C}[s]$ and an unstable region $\Lambda_2 \subseteq \mathbb{C}^r$:

   (i) For every $A$-module $M$ the canonical map $M_T \to M_{\mathfrak{T}}$ is an isomorphism and hence $M_T = M_{\mathfrak{T}}$ by identification.

   (ii) For every ideal $a$ of $A$ the reverse implication of (7.3) holds, i.e., for any ideal $a \subseteq A = \mathbb{C}[s]$ we get

   \[
   V(a) \cap \Lambda_2 = \emptyset \Rightarrow a \cap T \neq \emptyset.
   \]

   (iii) The preceding condition holds for all prime ideals of $\mathbb{C}[s]$. 
(iv) $\text{Max } (\mathbb{C}[s]_T) = \{ (m_s)_T; \lambda \in \Lambda_2 \}.$

If these conditions are satisfied the region $\Lambda_2$ is called ideal-convex.

2. If $\Lambda_2$ is ideal-convex and $B = \{ (y^0) \in F^{p+m}; P \circ y = Q \circ u \}$ is an IO behavior then $B$ is stable if and only if the matrix $P \in A^{k \times p}$ has a left inverse in $A^{p \times k}$.

Remark that the condition (ii) of the preceding theorem holds for all cyclic ideals $Af$ and especially for all prime ideals $Af$ of height one with irreducible $f$ since $V(f) \cap \Lambda_2 = \emptyset \iff f \in T$.

Lemma 7.3 The condition 1.(iii) of the preceding theorem holds for all maximal ideals $m_\mu$ of $\mathbb{C}[s]$ or, equivalently, for all Krull-zero-dimensional ideals $a$ (with finite $V(a)$) if and only if $\Lambda_1 = \bigcup_{t \in T} V(t)$, i.e., if $\Lambda_1$ is a (possibly infinite) union of hyper-surfaces.

This is the case if $\Lambda_2$ is a Cartesian product $\Lambda_2 = \prod_{\rho=1}^{r} \Lambda_{2,\rho}$. Indeed, for

$$
\mu \in \Lambda_1 \quad \text{and} \quad \mu_\rho \notin \Lambda_{2,\rho} : \quad t := s_\rho - \mu_\rho \in T \quad \text{and} \quad \mu \in V(t) \subset \Lambda_1.
$$

In dimension 2 this condition is equivalent with ideal-convexity since then all prime ideals are either principal or maximal.

Theorem and Definition 7.4 [43, Proposition 3.1.20 on p. 28] A compact subset $\Lambda_2 \subset \mathbb{C}^r$ is ideal-convex if it is polynomially convex [17, p. 53], i.e., if

$$
\Lambda_2 = \left\{ \lambda \in \mathbb{C}^r; \forall f \in \mathbb{C}[\lambda] : |f(\lambda)| \leq \max_{z \in \Lambda_2} |f(z)| \right\}.
$$

A compact set

$$
\Lambda_2 = \left\{ \lambda \in \mathbb{C}^r; \sup_{i=1, \ldots, l} |f_i(\lambda)| \leq 1 \right\}
$$

with $f_i \in \mathbb{C}[\lambda], i = 1, \ldots, l$, is polynomially convex and thus ideal-convex.

The proof of this theorem in [43] uses the Theorems A and B by Oka, Cartan, Serre on coherent analytic sheaves.

Examples 7.5 1. According to Theorem 7.4 the closed unit polydisc

$$
\{ z \in \mathbb{C}; |z| \leq 1 \}^r
$$

is ideal-convex. This is the standard example in the stabilization theory of discrete multidimensional input/output maps, compare [16] and [22].

2. Every unstable region in $\mathbb{C}$ is ideal-convex.

3. Lemma 7.3 implies that the two-dimensional unstable regions

$$
\{ z \in \mathbb{C}; \Re(z) \geq 0 \}^2 \quad \text{resp.} \quad \{ z \in \mathbb{C}; |z| \geq 1 \}^2
$$
are ideal-convex [51]. For higher dimensions this is an open problem.

4. Every compact polyrectangle
\[
\prod_{\rho=1}^r \{ \lambda_\rho \in \mathbb{C}; a_{\rho,1} \leq \Re(\lambda_\rho) \leq b_{\rho,1}, a_{\rho,2} \leq \Im(\lambda_\rho) \leq b_{\rho,2} \}
\]
with real numbers \(a_{\rho,i} \leq b_{\rho,i}, i = 1, 2\), is ideal-convex. The proof is analogous to that of Theorem 7.4.

5. For any stable region \(\Lambda_1 = \prod_{\rho=1}^r \Lambda_{1,\rho}\) as in Lemma 6.1 we have \(T = \mathbb{C} \setminus \{0\}\) and therefore \(\bigcup_{t \in T} V(t) = \emptyset\). Hence \(\Lambda_2\) is ideal-convex only if \(\Lambda_2 = \mathbb{C}^r\).

Open Problem 7.6 Decide ideal convexity for a given unstable region \(\Lambda_2\), for instance for the standard unstable regions
\[
\Lambda_2 := \{ z \in \mathbb{C}; \Re(z) \geq 0\}^r \text{ resp. } \Lambda_2 := \{ z \in \mathbb{C}; |z| \geq 1\}^r, r \geq 3
\]
and construct
\[
t \in a \cap T \quad \text{if} \quad V(a) \cap \Lambda_2 = \emptyset.
\]

8 Stabilization

Stabilization is defined according to section 2 with respect to the class of multidimensional IO systems from section 5 with the stable ones from section 6. In particular we require that the feedback system from equation (2.1) is well-posed. This signifies that the combined input \((u_2, u_1) \in \mathcal{F}^{p+m}\) is indeed an input according to section 5.

The assumptions and notations are those from sections 5, 6 and 7, in particular
\[
U_T = U_{st,T} = A_{1 \times k_{st}} R_{st}.
\]

Theorem 8.1 (Stabilizability and stabilization) [26, Theorems 4.4, 4.6, 5.8 and 5.14]

1. The following properties of an IO system \(B := \{ (y) \in \mathcal{F}^{p+m}; P \circ y = Q \circ u \}\) are equivalent:

   (i) The system is stabilizable.
   (ii) The submodule \(U_T \subseteq A_{1 \times (p+m)}\) is a direct summand.
   (iii) There is a matrix \(G_{st} \in A_{(p+m) \times k_{st}}\) such that \(R_{st} G_{st} R_{st} = R_{st}\).

Then \(E_1 := G_{st} R_{st} E_2^2\) is an idempotent matrix or projection in \(A_{(p+m) \times (p+m)}\) with \(U_T = A_{1 \times (p+m)} E_1\), and \(\tilde{U} := A_{1 \times (p+m)} (\text{id}_{p+m} - E_1)\) is a direct complement of \(U_T\).
2. If the equivalent conditions of 1. are satisfied there are bijections
\[ \left\{ X \in A_T^{(p+m) \times k_{st}} \ ; \ R_{st}X R_{st} = 0 \right\} / \left\{ X \in A_T^{(p+m) \times k_{st}} \ ; \ X R_{st} = 0 \right\} \cong \]
\[ \left\{ E \in A^{(p+m) \times (p+m)} \ ; \ E = E^2, \ U_T = A_T^{1 \times (p+m)} E \right\} \cong \]
\[ \left\{ \tilde{U} \leq A_T^{1 \times (p+m)} ; \ U_T \oplus \tilde{U} = A_T^{1 \times (p+m)} \right\} , \]
\[ \mathfrak{X} \mapsto E \mapsto \tilde{U}, \ E := E_1 + X R_{st}, \ \tilde{U} = A_T^{1 \times (p+m)} (id_{p+m} - E) . \]

Generically the matrix \((id_{p+m} - E)\left(\begin{array}{c} \lambda \\ \mu \end{array}\right)\) has rank \(m\). Generators of the first \(A_T\)-module in 2. can be computed via Gröbner bases.

3. Parametrization of stabilizing compensators: Let \(E = E^2\) be a matrix as in 2. such that \((id_{p+m} - E)_{(0 \ 1)}\) has rank \(m\) and let \(t \in T\) be a common denominator of the entries of \(E\). Define
\[ (-\tilde{Q}, \tilde{P}) := t(id_{p+m} - E) \in A^{(p+m) \times (p+m)} . \]

Then
\[ \tilde{B} := \left\{ \left(\begin{array}{c} \tilde{u} \\ \tilde{y} \end{array}\right) \in F^{p+m} ; \ \tilde{P} \circ \tilde{y} = \tilde{Q} \circ \tilde{u} \right\} \]
is an IO behavior with input \(\tilde{u}\) and a stabilizing compensator of the given behavior \(B\). It satisfies
\[ (\mathcal{B} \cap F_2^{p+m}) \bigoplus (\tilde{B} \cap F_2^{p+m}) = F_2^{p+m} = \bigoplus_{\lambda \in \Lambda_2} F_\lambda^{p+m} \text{ where} \]
\[ F_\lambda := \mathbb{C}[t]e^{\lambda t} \cong \bigoplus_{\mu \in \mathbb{N}^r} \mathbb{C} e_{\lambda, \mu} . \]

The steady state part \(\tilde{B} \cap F_2^{p+m}\) of each stabilizing compensator \(\tilde{B}\) of \(B\) is obtained in this fashion.

4. If the unstable region \(\Lambda_2\) is ideal-convex the behavior \(B\) is stabilizable if and only if the characteristic variety \(\text{char}(\mathcal{B})\) is contained in the stable region \(\Lambda_1\). Recall in this context that \(B\) is stable if and only if \(\text{char}(\mathcal{B}^0) \subseteq \Lambda_1\), and that \(\text{char}(\mathcal{B}) \subseteq \text{char}(\mathcal{B}^0)\).

5. If \(M_T\) is torsionfree the behavior \(B\) is stabilizable if and only if \(M_T \cong M\) (compare (7.2)) is a projective \(A_T\)-module. This is exactly the condition for the stabilizability of the transfer matrix \(H\) according to [33, Corollary 3].

**Algorithm 8.2** 1. At various instances of the stabilization theory one has to check whether a variety meets \(\Lambda_2\). All relevant unstable regions \(\Lambda_2 \subseteq \mathbb{C}^r = \mathbb{R}^{2r}\) are real semi-algebraic [5, p. 21]. For these the equation \(V(a) \cap \Lambda_2 = \emptyset\) can be constructively decided by means of the Tarski-Seidenberg-principle and cylindrical algebraic decomposition. This fact is of highest significance for stabilization theory. We learned it from S. Tsarev at the D2-conference of the Gröbner semester in Linz and have yet to study.
it in detail. In the mean-time we realized that the application of such methods in stabilization theory has already a longer history, for instance in several papers by N. K. Bose et al. [1, 6, 8, 24].

Whether under the necessary condition \( V(a) \cap \Lambda_2 = \emptyset \) the inequality \( a \cap T \neq \emptyset \) can likewise be decided and whether even a particular element \( t \in a \cap T \) can be constructed is open in the moment. This problem is obviously related to the Open Problem 7.6 concerning ideal-convexity and was also discussed in [52] for the closed unit polydisc.

2. The annihilator ideal \( a := \text{ann}_A(M^0) \) (compare (5.1)) can be computed via Gröbner bases. Its variety \( V(a) = \text{char}(M^0) = \text{char}(B^0) \) is the characteristic variety of the autonomous part \( B^0 \) of \( B \). According to Theorem 6.2 the behavior \( B \) is stable if and only if \( V(a) \cap \Lambda_2 = \emptyset \), and this can be decided due to item 1.

3. Stabilizability is checked via the condition 1.(iii) of Theorem 8.1, i.e., by solving the inhomogeneous linear equation

\[
R_{st}G_{st}R_{st} = R_{st} \quad \text{in} \quad A_T^{(p+m) \times k_{st}}. \tag{8.1}
\]

In particular, one needs the matrix \( R_{st} \) which is known in the following case. The annihilator \( a := \text{ann}_A(t(M)) \) of the torsion module \( t(M) \) of \( M \) can be computed via Gröbner bases and the equation \( V(a) \cap \Lambda_2 = \emptyset \) can be checked constructively according to item 1. If this is the case Theorem 7.1 implies \( R_{st} = R_{cont} \) and therefore \( R_{st} \) is known in this important case.

4. Assume, more generally, that a matrix \( R_{st} \in A_{k_{st} \times (p+m)}^{k_{st}} \) is known such that \( U_T = A_T^{k_{st} \times k_{st}}R_{st} \). Recall that the matrix \( G_{st} \) in equation (8.1) has denominators in \( T \). Instead of solving this inhomogeneous equation we compute generators of the solution module \( L \) of the homogeneous linear system \( R_{st}GR_{st} = xR_{st} \), i.e.,

\[
L := \{(G, x) \in A_{k_{st} \times (p+m)}^{k_{st}} \times A; \ R_{st}GR_{st} - xR_{st} = 0\},
\]

via Gröbner bases for the polynomial algebra \( A = \mathbb{C}[s] \) and then obtain generators of the ideal

\[
a := \text{proj}(L) = \{x \in A; \exists (G, x) \in L\}.
\]

The equation (8.1) has a solution

\[
G_{st} \in A_T^{(p+m) \times k_{st}} \quad \text{if and only if there is} \quad t \in a \cap T, \quad \text{and then}
\]

\[
G_{st} := \frac{G}{L} \quad \text{with} \quad (G, t) \in L \quad \text{is one solution.}
\]

Recall that for the existence of \( t \in a \cap T \) the equation \( V(a) \cap \Lambda_2 = \emptyset \) is necessary and can be checked constructively according to item 1. The decision of \( a \cap T \neq \emptyset \) and the eventual construction of \( t \in a \cap T \) are, however, still open as explained in item 1.

5. If one solution \( G_{st} \) and its ensuing idempotent matrix

\[
E_{1} := G_{st}R_{st} \in A_T^{(p+m) \times (p+m)}
\]
have been computed the construction of all other matrices $E$ according to Theorem 8.1, (2), is simple. Via Gröbner bases one computes $A$-generators of the solution module

$$L_0 := \{ X \in A^{(p+m) \times k_{st}} ; R_{st}XR_{st} = 0 \}$$

and these are also $A_T$-generators of the solution module

$$L_{st} := \{ X \in A^{(p+m) \times k_{st}} ; R_{st}XR_{st} = 0 \} = L_{0,T}$$

which appears in the theorem.

6. The preceding considerations show that important parts of the algorithms are constructive, but that difficult open problems remain for the constructive checking of stabilizability of a given system. The construction of all stabilizing compensators is then simple.

9 Proper stabilization

As explained in the introduction with many references most papers on multidimensional stabilization treat the stabilization of discrete proper transfer matrices. The main advantage of the latter is that they can often be naturally interpreted as input/output maps or operators, see, for instance, [10, chapter 1 and 3] and [27, §6]. In this section we characterize those IO systems which can be stabilized according to the preceding section with the additional property that the resulting feedback system is proper (and, of course, stable).

The assumptions and notations of the previous sections remain in force. First we will define the ring of proper and stable rational functions which plays the part of the ring $A_T$ of stable rational functions in the previous sections.

**Definition and Corollary 9.1** [27, Theorem 6.50] Let $\deg_{s_r}(f)$ denote the univariate degree of $f \in C[s] = C[s_1, \ldots, \hat{s}_r, \ldots, s_r]$. The multivariate or componentwise degree of a polynomial is the function

$$\deg : C[s] \setminus \{0\} \to N^r, \ f \mapsto (\deg_{s_1}(f), \ldots, \deg_{s_r}(f)).$$

It satisfies $\deg(fg) = \deg(f) + \deg(g)$ and can therefore be extended to the field of rational functions $C(s) := \{ \frac{a}{b}, \ a, b \in C[s], \ b \neq 0 \}$ via

$$\deg : C(s) \setminus \{0\} \to Z^r, \ \frac{a}{b} \mapsto \deg \left( \frac{a}{b} \right) := \deg(a) - \deg(b).$$

In contrast to the degree-function induced by a term well-order on $N^r$ the componentwise degree $\deg(f)$ of a polynomial $f = \sum_{\mu \in N^r} a_\mu s_\mu \in C[s]$ does not, in general, belong to its support $\text{supp}(f) := \{ \mu \in N^r ; \ a_\mu \neq 0 \}$, for instance

$$\deg(s_1 + s_2) = (1,1) \notin \text{supp}(s_1 + s_2) = \{(1,0), (0,1)\}.$$

The polynomials satisfying $\deg(f) \in \text{supp}(f)$ are called componentwise unital (cw-unital).
Theorem and Definition 9.2 [27, Theorem 6.60] A rational function in \( \mathbb{C}(s) \) is called proper if it is a power series in \( s^{-1} := (s_1^{-1}, \ldots, s_r^{-1}) \). Thus we define the algebra of proper rational functions
\[
\mathcal{P} := \mathbb{C}(s) \cap \mathbb{C}[s^{-1}] .
\] (9.1)
It can also be described as
\[
\mathcal{P} = \left\{ \frac{a}{t} \in \mathbb{C}(s); \ a, t \in \mathbb{C}[s], \ t \text{ is cw-unital}, \ \deg(a) \leq_{cw} \deg(t) \right\} .
\] (9.2)
where \( \leq_{cw} \) is the partial ordering of \( \mathbb{Z}^r \) defined by
\[
\mu \leq_{cw} \nu : \iff \forall \rho = 1, \ldots, r : \mu_\rho \leq \nu_\rho : \iff \exists \gamma \in \mathbb{N}^r : \nu = \mu + \gamma .
\]
Then
\[
\mathcal{S} := \mathbb{C}[s]_T \cap \mathcal{P} = \mathbb{C}[s]_T \cap \mathbb{C}[s^{-1}]
\] (9.3)
is the algebra of proper stable rational functions.

For the interpretation of a proper transfer matrix as discrete IO operator we identify
\[
\mathbb{C}[s^{-1}] = \mathbb{C}^{N_r}, \ u = (u_\mu)_\mu \in \mathbb{N}^r = \sum_{\mu \in \mathbb{N}^r} u_\mu s^{-\mu} .
\]
The action of \( \mathbb{C}[s] \) on \( \mathbb{C}^{N_r} \) is that by left shifts, i.e., \( (s^h \circ u)_\mu = u_{\lambda + \mu} \). The multiplication or convolution \( * \) of \( \mathbb{C}[s^{-1}] \) induces the action on itself by right shifts. Then [27, (6.37)]
\[
P \circ (H \ast u) = Q \circ u \quad \text{for} \quad P, Q \in \mathbb{C}[s] , \ u \in \mathbb{C}^{N_r} \quad \text{if} \quad H := P^{-1} Q \in \mathcal{P} .
\]
In particular, the unit impulse \( \delta_0 = 1 = s^{-0} \) gives rise to the impulse response \( H = H \ast \delta_0 \). More generally, a proper transfer matrix \( H \) of a discrete IO system \( \mathcal{B} \) can be interpreted as the IO map \( u \mapsto H \ast u \) and as its impulse response, in detail
\[
\mathcal{B} = \left\{ \begin{pmatrix} y \\ u \end{pmatrix} \in (\mathbb{C}^{N_r})^{p+m} ; \ P \circ y = Q \circ u \right\} , \ \text{rank}(P) = p , \ PH = Q , \ H \in \mathcal{P}^{p \times m} .
\]
\[
\begin{pmatrix} H \ast u \\ u \end{pmatrix} \in \mathcal{B} \quad \text{for all} \quad u \in (\mathbb{C}^{N_r})^m , \quad \text{especially} \quad H_{-j} = H \ast (1, \ldots, 0, 1, 0, \ldots, 0)^\top .
\]
The following weak, but important condition on \( \mathbb{C}^r = \Lambda_1 \uplus \Lambda_2 \) is satisfied by all standard stability decompositions, for example those in (6.2). Let
\[
\text{proj}_\rho : \mathbb{C}^r \rightarrow \mathbb{C} , \ \lambda \mapsto \lambda_\rho ,
\]
denote the projection on the \( \rho^{th} \) component for \( \rho = 1, \ldots, r \). In the sequel we assume
\[
\text{proj}_\rho(A_2) \subseteq \mathbb{C} , \ \text{i.e.,} \ \mathbb{C} \setminus \text{proj}_\rho(A_2) \neq \emptyset , \ \text{for all} \quad \rho = 1, \ldots, r ,
\]
choose \( \alpha \in \prod_{\rho=1}^r (\mathbb{C} \setminus \text{proj}_\rho(A_2)) \) and define
\[
p_\rho := s_\rho - \alpha_\rho , \quad p := (p_1, \ldots, p_r) = s - \alpha \in \mathbb{C}[s]^r , \quad \text{hence} \quad p_\rho^{-1} \in \mathcal{S} .
\]
Theorem 9.3 We give two descriptions of the ring of proper and stable rational functions:

1. \( S = \{ \frac{a}{t} \in \mathbb{C}(s); \ a \in \mathbb{C}[s], \ t \in T \text{ and } \text{cw-unital}, \ \deg (\frac{a}{t}) \leq \text{cw} \ 0 \} \). This description is derived from that of \( \mathcal{P} \) in equation (9.2).

2. \( S_1 := \mathbb{C}[-1] := \mathbb{C}\left[\frac{1}{s_1 - \alpha_1}, \ldots, \frac{1}{s_r - \alpha_r}\right] = \{ \frac{a}{p_1^{\mu}}, \ a \in \mathbb{C}[p], \ \deg(a) \leq \text{cw} \mu \} \) is the polynomial algebra in \( p_{-1}, \ldots, p_{-r} \) and a subring of \( S \) where
   \[ \mathbb{C}[p] := \mathbb{C}[p_1, \ldots, p_r] = \mathbb{C}[s - \alpha] = \mathbb{C}[s], \quad \text{but} \]
   \[ \mathbb{C}[p^{-1}] \neq \mathbb{C}[s^{-1}] \quad \text{since, for instance,} \quad p_{-1}^{-1} = \sum_{k=0}^{\infty} \alpha_k (s_1^{-1})^{k+1} \notin \mathbb{C}[s^{-1}]. \]

3. The set \( T_1 := \left\{ \frac{t}{p_{-r}(T)}; \ t \in T \text{ and } \text{cw-unital} \right\} \) is a multiplicatively closed saturated subset of \( S_1 \) and \( S = S_1, T_1 \) is the associated quotient ring.

Theorem 9.4 As a quotient ring of the polynomial algebra \( S_1 = \mathbb{C}[p^{-1}] \) the ring \( S \) of proper stable rational functions is a Noetherian factorial integral domain. Its prime elements are the irreducible polynomials in \( S_1 \setminus T_1 \).

With these preparations we can now treat the proper stabilizability of IO systems.

Definition and Corollary 9.5 An IO system \( \mathcal{B} \) as in section 5 is called proper if its transfer matrix \( H \) belongs to \( \mathcal{P}^{p \times m} \). If it is proper and stable according to section 6 then \( H \in S^{p \times m} \).

The following theorem is the analogue of Theorem 8.1 for proper stabilization.

Theorem and Definition 9.6 (Proper stabilizability) With the notations from Theorem 8.1 the following properties are equivalent for an IO system
\[ \mathcal{B} = \left\{ \left( \begin{array}{c} y \\ u \end{array} \right) \in \mathbb{F}^{p+m}, \ P \circ y = Q \circ u \right\} : \]

1. The system \( \mathcal{B} \) is proper stabilizable, i.e., it is stabilizable in the sense of Definition 2.1 and section 8 with the additional property that the resulting feedback system is proper.

2. The \( S \)-module \( U \cap S^1 \times I \) is a direct summand of \( S^1 \times I \).

Algorithm 9.7 We use the preceding theorem to check algorithmically whether an IO behaviour \( \mathcal{B} \) is proper stabilizable and, if so, to construct all compensators. This algorithm is subdivided into four main steps:

1. Find \( \mathbb{C}[s] \)-generators of \( U_u \): In many cases this can be solved according to Algorithm 8.2, item 3.
2. Find an $S$-generating system of $U_{\bar{T}} \cap S^{1 \times t}$, $U_{\bar{T}} = U_{st,T}$: Recall that

$$T = \bigcap_{\lambda \in \Lambda_2} (\mathbb{C}[s] \setminus \mathfrak{m}_\lambda)$$

and that therefore

$$U_{st} = \mathbb{C}[s]^{1 \times t} \bigcap \bigcap_{\lambda \in \Lambda_2} U_{\mathfrak{m}_\lambda}$$

is $T$-closed, i.e.,

$$u \in \mathbb{C}[s]^{1 \times t}, \ t \in T, \ tu \in U_{st} \implies u \in U_{st}.$$  

This property implies the quotient representation

$$U_{st,T} \cap S^{1 \times t} = \hat{U}_{T_1}$$

where

$$\hat{U} := \mathbb{C}[p^{-1}]^{1 \times t} \cap \mathbb{C}[p,p^{-1}] U_{st} \subseteq \mathbb{C}[p^{-1}]^{1 \times t}.$$  

Hence $\mathbb{C}[p^{-1}]$-generators of $\hat{U}$ are also $S$-generators of $U_{st,T} \cap S^{1 \times t}$. These can be computed by means of the following argument from [55]. Let

$$\mathbb{C}[p,q] := \mathbb{C}[p_1, \ldots, p_r, q_1, \ldots, q_r]$$

be the polynomial ring in $2r$ indeterminates with the ideal

$$I := \sum_{\rho=1}^{r} \mathbb{C}[p,q](p_\rho q_\rho - 1).$$

The algebra isomorphism

$$\mathbb{C}[p,p^{-1}] \cong \mathbb{C}[p,q]/I, \ p_\rho \leftrightarrow p_\rho^{-1}, \ q_\rho \leftrightarrow q_\rho^{-1},$$

makes the algebra $\mathbb{C}[p,p^{-1}]$ of Laurent polynomials accessible to the Gröbner calculus. We compute a Gröbner basis of $\mathbb{C}[p,q] U_{st} + I \mathbb{C}[p,q]^{1 \times t}$ with respect to an elimination term order for $p_1, \ldots, p_r$. With the identification $\mathbb{C}[q] = \mathbb{C}[\bar{q}] = \mathbb{C}[p^{-1}]$ its elements in $\mathbb{C}[q]^{1 \times t}$ are a Gröbner basis and hence a generating system of $\hat{U}$ and of $U_{T_1} \cap S^{1 \times t}$.

3. Check whether $U_{\bar{T}} \cap S^{1 \times t}$, $U_{\bar{T}} = U_{st,T}$, is a direct summand of $S^{1 \times t}$: Part 4 of Algorithm 8.2 can be applied in this situation with the polynomial algebra $\mathbb{C}[p^{-1}]$ and $U_{\bar{T}} \cap S^{1 \times t}$ instead of $\mathbb{C}[s]$ and $U_{\bar{T}}$. Notice, however, that part 4 of Algorithm 8.2 requires the algorithmic construction of $t_1 \in T_1 \cap a_1$ for an ideal $a_1 \subseteq \mathbb{C}[p^{-1}]$ which is an open problem in general.

4. Construct the stabilizing compensators: Assume that the IO behavior $\mathcal{B}$ is proper stabilizable. According to parts 4 and 5 of Algorithm 8.2 we construct one and then all direct complements

$$V_2 = S^{1 \times k_2} R_2^0, \ R_2^0 = (-Q_2^0, P_2^0) \in S^{k_2 \times (p+m)}, \mbox{ of } U_{\bar{T}} \cap S^{1 \times t} \mbox{ in } S^{1 \times t}.$$
Let $t \in T$ be a common denominator of the entries of $R^2$, i.e.,

$$(Q_2, P_2) := t(-Q_2^s, P_2^s) \in \mathbb{C}[s]^{k_2 \times (p+m)}.$$ 

Then generically

$$\text{rank}(P_2) = m$$

and $B^2 := \{(u_2, y_2) \in \mathbb{F}^{p+m}; P_2 \circ y_2 = Q_2 \circ u_2\}$ is a proper stabilizing compensator of $B$. All such compensators are obtained in the same fashion.

10 Discrete BIBO stability

In this section we discuss stability in the discrete case for the signal space $\mathbb{C}^{N^r}$ and the stability decomposition

$$\mathbb{C}^r = \Lambda_1 \cup \Lambda_2, \quad \Lambda_2 := \{w \in \mathbb{C}; |w| \geq 1\}^r.$$  \hspace{1cm} (10.1)

Then the ring $S$ of proper stable rational functions in the sense of (9.3) coincides precisely with ring of causal structurally stable rational functions as discussed, for instance, in [16, Definition 3.47] and [22, p. 60]. This implies in particular that the transfer matrix of a proper stable system is BIBO stable.

To see this in detail we need some preparations. The notations are the same as in section 9. We use various rings of power series and the abbreviation

$$z_\rho := s_\rho^{-1}, \text{ hence } \mathbb{C}[s^{-1}] = \mathbb{C}[z] = \mathbb{C}[z_1, \ldots, z_r] = \mathbb{C}^{N^r}.$$ 

For a vector $R := (R_1, \ldots, R_r) \in \mathbb{R}^{\geq 0}$, i.e., $R_\rho > 0$ for all $\rho$, we consider the open resp. closed polydisc

$$U(R) := \prod_{\rho=1}^{r} \{z_\rho \in \mathbb{C}; |z_\rho| < R_\rho\} \subset U(R) := \prod_{\rho=1}^{r} \{z_\rho \in \mathbb{C}; |z_\rho| \leq R_\rho\}$$

in particular the unit polydiscs $U(1) \subset U(1)$ with $1 := (1, \ldots, 1)$.

If $X$ is an arbitrary subset of $\mathbb{C}^r$ a function is called holomorphic in $X$ if it is defined in an open neighborhood of $X$ and is there holomorphic. If $X$ is open $f$ need, of course, only be defined on $X$. The algebra $O(X)$ of all holomorphic functions on $X$ is a subalgebra of the algebra $\mathbb{C}^0(X)$ of all continuous functions. If $f$ is holomorphic in $U(R)$, $R \in \mathbb{R}^{\geq 0}$, it has a Taylor series expansion $f(z) = \sum_{\mu \in \mathbb{N}^r} a_\mu z^\mu$ which is compactly convergent in $U(R)$, i.e., uniformly convergent on compact sets of $U(R)$.

The spaces of power series

$$B_R := \left\{ f = \sum_{\mu \in \mathbb{N}^r} a_\mu z^\mu \in \mathbb{C}[z]; ||f||_R := \sum_{\mu \in \mathbb{N}^r} |a_\mu| R^\mu < \infty \right\}$$  \hspace{1cm} (10.2)
and in particular

\[ \ell^1(N^r) := B_1 = \left\{ f = \sum_{\mu \in N^r} a_\mu z^\mu; \| f \|_1 = \sum_{\mu \in N^r} |a_\mu| < \infty \right\} \]

are subalgebras of \( \mathbb{C}[[z]] \) and indeed Banach algebras with respect to \( \| f \|_R \) [15, Satz 1 on p. 16], and moreover

\[ B_R \subset O(U(R)) \cap C^0(U(R)). \]

Let \( \ell^\infty(N^r) \) denote the space of all bounded multisequences, i.e.,

\[ \ell^\infty(N^r) \ := \ \left\{ u = (u_\mu)_{\mu \in N^r} = \sum_{\mu \in N^r} u_\mu z^\mu \in \mathbb{C}^{N^r}; \ \exists M > 0: \forall \mu \in N^r : |u_\mu| \leq M \right\}. \]

**Corollary 10.1** The algebra \( B_1 = \ell^1(N^r) \) and the space \( \ell^\infty(N^r) \) are \( B_1 \)-submodules of \( \mathbb{C}[[z]] \) with respect to the convolution multiplication of \( \mathbb{C}[[z]] \). In other terms,

if \( H \in \ell^1(N^r), \ u = (u_\mu)_{\mu \in N^r} \in \ell^\infty(N^r) \) then \( H * u \in \ell^\infty(N^r), \ i = 1, \infty. \)

The first resp. the second of these properties are called the \( \ell^1, \ell^1 \)- resp. BIBO stability of the transfer function \( H \).

Let now \( S_{ss} \) denote the ring of causal structurally stable rational functions according to [16, Definition 3.47], i.e.,

\[ S_{ss} := \left\{ \frac{a(z)}{b(z)} \in \mathbb{C}(z) = \mathbb{C}(s); \ \forall z \in U(1) : b(z) \neq 0 \right\} \subset \mathbb{C}[z]_m = \mathbb{C}(z) \cap \mathbb{C}[[z]] \]

where \( m \) is the maximal ideal \( m := \sum_{\mu=1}^r \mathbb{C}[z]z_\mu \subset \mathbb{C}[z] \), and let

\[ S := \mathcal{P} \cap \mathbb{C}[s]_T = \mathbb{C}[z]_m \cap \mathbb{C}[s]_T \subset \mathbb{C}(s) = \mathbb{C}(z) \]

be the ring of proper stable rational functions with respect to the stability decomposition (10.1) from (9.3).

**Theorem 10.2** These two rings of stable rational functions coincide, i.e.,

\[ S = S_{ss}, \quad \text{in particular} \quad S \subset O(U(1)) \subset B_1 = \ell^1(N^r). \]

Therefore proper stable rational functions are \( \ell^1, \ell^1 \)-stable and BIBO stable.

The proof of the preceding theorem uses multivariate Laurent series [50, p. 38ff]. In [26, Remark 5.12, (1)] we called it an open problem.
Remark 10.3  A causal structurally stable rational function

\[ H = \frac{a(z)}{b(z)} \quad \text{with} \quad b(z) \neq 0 \quad \text{for all} \quad z \in \overline{U}(1) \quad \text{and relatively prime} \quad a, b \]

belongs to \( B_1 = \ell^1(\mathbb{N}^r) \subset C(U(1)) \cap C^0(U(1)) \), i. e., \( H \) is BIBO stable. If \( a = 1 \) the converse is true as is easily seen. In general, BIBO stability of \( H \) implies that \( b(z) \) has no zeros in \( U(1) \) and that any zero of \( b \) in \( \overline{U}(1) \) is also one of \( a \). It may happen that there are infinitely many such zeros in \( \overline{U}(1) \setminus U(1) \).

Consult [10, section 1.2, Problem 1 on p. 245], [8, section 1.4.1], [11, pp. 190–198], [21], [47, pp. 492–496] and the original papers quoted there for further discussions of multidimensional discrete BIBO stable transfer functions and various stability tests.

11 Continuous BIBO stability

In this section we discuss BIBO stability in the continuous case of partial differential equations for various signal spaces and the stability decomposition

\[ C' = \Lambda_1 \cup \Lambda_2; \quad \Lambda_2 := \{ w \in \mathbb{C}; \Re(w) \geq 0 \}^r. \]

Stability of polynomials, rational functions, the ring \( S \) etc. always refer to this decomposition. In [7, p. 12], [21, Definition 5 on p. 140] and [35, p. 79] the stable polynomials in our sense are called strict Hurwitz polynomials.

In the last chapter 8 on ‘Stability of Multidimensional Continuous Systems’ of the survey paper [21] E. I. Jury conjectures with an algebraic, non-analytic reduction to the discrete case that the inverse of a bivariate very strict Hurwitz polynomial (VSHP) is BIBO stable in the sense that its impulse response is an absolutely integrable function [21, Definition 6, Theorem 23 and Remark]. However, the exact nature of the continuous systems is not specified and the (usually difficult) existence of the impulse response is not addressed in that survey. In [35, Theorem 6.1] Jury’s conjecture is quoted as a theorem. Notice that there are other types of continuous systems than those of the present paper, for instance delay-differential systems.

In this section we show that the impulse response of a proper rational function exists, but is a measure and not a function. We derive necessary and sufficient conditions for BIBO stability and in particular prove Jury’s conjecture.

If \( X \) is a set,

\[ x = (x_1, \ldots, x_r) \in X^r \quad \text{and} \quad S \subseteq \{1, \ldots, r\}, \quad S' := \{1, \ldots, r\} \setminus S \]

we define the subvector \( x_S := (x_\rho)_{\rho \in S} \in X^S \), for instance

\[ \mu = (\mu_1, \ldots, \mu_r), \quad 1 := (1, \ldots, 1) \in \mathbb{N}^r, \quad \mu_S = (\mu_\rho)_{\rho \in S}, \quad 1_S \in \mathbb{N}^S, \]

and identify \( x = (x_S, x_{S'}) \in X^r = X^S \times X^{S'} \). As before we use the variables

\[ s = (s_1, \ldots, s_r), \quad z = (z_1, \ldots, z_r) := s^{-1}, \quad z_\rho := s_\rho^{-1}, \quad t = (t_1, \ldots, t_r) \]
which we consider both as indeterminates and as vectors in \( C^r \).

We need certain preparations concerning distributions from [40, pp. 170–180] and consider the subspace \( \mathcal{D}'_+ := \mathcal{D}'_+(\mathbb{R}^r) \subset \mathcal{D}' := \mathcal{D}'(\mathbb{R}^r) \) of all distributions with left bounded support, i.e., whose support is contained in a subset \( t + \mathbb{R}_{\geq 0}^r, t \in \mathbb{R}^r \). The space \( \mathcal{D}'_+ \) is a commutative integral domain with respect to the convolution product \( * \) [40, Theorem XIV on p. 173]. If \( \varphi_1, \ldots, \varphi_r \) are univariate functions in \( \mathcal{D}(\mathbb{R}) = C^\infty_c(\mathbb{R}) \) and \( T_1, \ldots, T_r \) are univariate distributions in \( \mathcal{D}'(\mathbb{R}) \), then their tensor products are defined [40, chapter IV]:

\[
T := \otimes_{\rho=1}^r T_{\rho} \in \mathcal{D}'(\mathbb{R}^r), \quad \varphi := \otimes_{\rho=1}^r \varphi_{\rho} \in \mathcal{D}(\mathbb{R}^r),
\]

\[
\varphi(t_1, \ldots, t_r) := \prod_{\rho=1}^r \varphi_{\rho}(t_{\rho}), \quad T(\varphi) := \prod_{\rho=1}^r T_{\rho}(\varphi_{\rho}).
\]

If the \( T_{\rho} \) have even left bounded support, i.e., are contained in \( \mathcal{D}'_+(\mathbb{R}) \), then so has their tensor product. With the Heaviside step function in one variable \( t_1 \in \mathbb{R} \)

\[
Y(t_1) := \begin{cases} 1 & \text{if } t_1 > 0 \\ 0 & \text{otherwise} \end{cases}
\]

and its derivative \( \delta := Y' \), \( \delta(\varphi) = \varphi(0) \), we define [40, (II.2.31)]

\[
Y_k(t_1) := \begin{cases} \frac{t_1^{k-1}}{(k-1)!} Y(t_1) & \text{if } k \in \mathbb{Z}, k > 0 \\ \delta(-k) = s_1^{-k} \delta & \text{if } k \leq 0 \end{cases}, \quad Y_k \in \mathcal{D}'_+(\mathbb{R}),
\]

where \( f(s) \circ \delta \) denotes the action of a polynomial as differential operator. For \( k > 0 \) the distribution \( Y_k \) is a function with

\[
Y_k(\varphi) = \int_0^\infty t_1^{k-1} \frac{1}{(k-1)!} \varphi(t_1) dt_1, \quad \varphi \in \mathcal{D}(\mathbb{R}).
\]

The distributions \( Y_k \) are defined for all complex numbers \( k \), but we do not need this here. The \( Y_k \) satisfy [40, p. 43, (VI.5.6)]

\[
Y_k^l = s_1 \circ Y_k = Y_{k-l} \quad \text{and} \quad Y_k \ast Y_l = Y_{k+l}, \quad k, l \in \mathbb{Z}.
\]

These univariate distributions give rise to their \( r \)-dimensional counter-parts

\[
Y_\mu := Y_{\mu_1} \ast \ldots \ast Y_{\mu_r} \in \mathcal{D}'_+(\mathbb{R}^r), \quad \mu \in \mathbb{Z}^r,
\]

in particular for \( \mu \in \mathbb{N}^r \), \( \varphi \in \mathcal{D}(\mathbb{R}^r) \) and \( S := S(\mu) := \operatorname{supp}(\mu) = \{ \rho; \mu_{\rho} \neq 0 \} \):

\[
Y_\mu = \frac{t_{S(\mu)-1_{S(\mu)}}^{\mu_S}}{(\mu_S - 1_{S(\mu)})!} \otimes_{\rho \in S(\mu)} Y(t_{\rho}) \otimes_{\rho \in S(\mu)} \delta(t_{\rho})
\]

\[
Y_\mu(\varphi) = \int_{0_S}^{\infty} t_{S-1_{S}}^{\mu_S-1_{S}} \varphi(t_S, 0_S) dt_S,
\]
where
\[ t^{\mu - 1}_{S} = \prod_{\rho \in S} t^{\mu_{\rho} - 1}, \quad (\mu S - 1)_{S} = \prod_{\rho \in S} (\mu_{\rho} - 1)_{\rho}, \]
\[ \int_{0}^{\infty} (-) d t_{S} := \int_{0}^{\infty} \cdots \int_{0}^{\infty} (-) \prod_{\rho \in S} d t_{\rho}. \]
It is easy to see that the identities
\[ Y_{\mu} \ast Y_{\nu} = Y_{\mu + \nu} \quad \text{for} \quad \mu, \nu \in \mathbb{Z}^{r} \]
\[ s^{\nu} \circ Y_{\mu} = Y_{-\nu} \ast Y_{\mu} = Y_{\mu - \nu} \quad \text{for} \quad \mu \in \mathbb{Z}^{r}, \nu \in \mathbb{N}^{r} \]
hold. With \( B_{R} \) from (10.2) let
\[ \mathbb{C}(z) := \bigcup_{R \in \mathbb{R}^{\geq 0}} B_{R} \subset \mathbb{C}[z] \]
denote the algebra of \textit{(locally) convergent} power series. We also consider the algebras of \textit{Laurent} polynomials resp. power series
\[ \mathbb{C}[z]_{z} = \bigoplus_{\mu \in \mathbb{Z}^{r}} \mathbb{C}z^{\mu}, \quad \mathbb{C}[z][z] = \{ s^{\mu} H(z); \ H \in \mathbb{C}[z], \mu \in \mathbb{N}^{r} \}, \]
\[ \mathbb{C}(z)_{z} = \{ s^{\mu} H(z); \ H \in \mathbb{C}(z), \mu \in \mathbb{N}^{r} \}. \]
The equations \( Y_{\mu} \ast Y_{\nu} = Y_{\mu + \nu} \) imply that the map
\[ \mathbb{C}[z]_{z} \to \mathcal{D}_{+}^{r}(\mathbb{R}^{r}), \quad f = \sum_{\mu \in \mathbb{Z}^{r}} f_{\mu} z^{\mu} \mapsto f(Y) := \sum_{\mu \in \mathbb{Z}^{r}} f_{\mu} Y_{\mu}, \]
is an algebra homomorphism, and it is indeed a monomorphism. This monomorphism can be extended to \( \mathbb{C}(z)_{z} \) as follows. A convergent power series \( H = \sum_{\mu \in \mathbb{N}^{r}} H_{\mu} z^{\mu} \) can be considered as an analytic functional on \( \mathbb{C}^{r} \) [17, chapter 4, §5], [27, pp. 64–70], i.e., as the continuous linear map
\[ H : \mathcal{O}(\mathbb{C}^{r}) \to \mathbb{C}, \quad u = \sum_{\mu \in \mathbb{N}^{r}} u_{\mu} z^{\mu} \mapsto H(u) := \sum_{\mu \in \mathbb{N}^{r}} H_{\mu} u_{\mu}, \]
where \( \mathcal{O}(\mathbb{C}^{r}) := \bigcap_{R \in \mathbb{R}^{\geq 0}} B_{R} \subset \mathbb{C}(z) = \bigcup_{R \in \mathbb{R}^{\geq 0}} B_{R} \)
is the algebra of entire functions or everywhere convergent power series in \( z \). Its \textit{Laplace transform} [17, Definition 4.5.2]
\[ \hat{H}(t) := H(z(e^{t} \cdot t)) = \sum_{\mu \in \mathbb{N}^{r}} H_{\mu} \frac{t^{\mu}}{\mu!} \]
is contained in \( \mathcal{O}(\mathbb{C}^{r}, \exp) \), i.e., is an entire holomorphic function of \textit{exponential type}, and the map
\[ \mathbb{C}(z) \cong \mathcal{O}(\mathbb{C}^{r}, \exp), \quad H = \sum_{\mu \in \mathbb{N}^{r}} H_{\mu} z^{\mu} \mapsto \hat{H}(t) = H(z(e^{t} \cdot t)) = \sum_{\mu \in \mathbb{N}^{r}} H_{\mu} \frac{t^{\mu}}{\mu!}, \]
is an isomorphism [27, (4.28)].
Theorem 11.1 The algebra monomorphism \( \mathbb{C}[z] \rightarrow \mathcal{D}'_+, \ z^\mu \mapsto Y_\mu \), can be uniquely extended to algebra monomorphisms

\[
\mathbb{C}(z) \rightarrow \mathcal{D}'_+, \quad H = \sum_{\mu \in \mathbb{N}^r} H_\mu z^\mu \mapsto H(Y) := \sum_{\mu \in \mathbb{N}^r} H_\mu Y_\mu, \quad \text{and}
\]

\[
\mathbb{C}(z)_2 \rightarrow \mathcal{D}'_+, \quad a = s^\mu H \mapsto a(Y) := Y_{-\mu} \ast H(Y) = s^\mu \circ H(Y), \quad \mu \in \mathbb{N}^r,
\]

the first one being continuous in the sense that it transforms analytically convergent sequences in \( \mathbb{C}(z) \) into convergent sequences of distributions.

The algebra homomorphism \( \mathbb{C}(z)_2 \rightarrow \mathcal{D}'_+(\mathbb{R}^r) \) makes \( \mathcal{D}'_+(\mathbb{R}^r) \) a \( \mathbb{C}(z)_2 \)-module via

\[
a \circ T := a(Y) \ast T.
\]

Each \( H(Y), \ H \in \mathbb{C}(z), \) is indeed a measure, i.e., a continuous linear function on \( \mathcal{C}_0^r(\mathbb{R}^r) \). If \( \varphi(t) \) is a continuous function with compact support in \( K := \{ t \in \mathbb{R}^r; \forall \rho : -R_\rho \leq t \leq R_\rho \}, \ R \in \mathbb{R}_{>0}^r, \)

and \( \| \varphi \|_K := \sup_{t \in K} |\varphi(t)| \) is its maximum norm then

\[
\left| \sum_{\mu \in \mathbb{N}^r} H_\mu Y_\mu(\varphi) \right| \leq \sum_{\mu \in \mathbb{N}^r} |H_\mu| \frac{R_\mu}{\mu!} \| \varphi \|_K,
\]

where \( \sum_{\mu \in \mathbb{N}^r} |H_\mu| \frac{R_\mu}{\mu!} < \infty \) since \( \hat{H}(t) = \sum_{\mu \in \mathbb{N}^r} H_\mu \frac{t^\mu}{\mu!} \) is entire. Here a sequence of (locally) convergent power series is analytically convergent [15, p. 31] if it is convergent in some Banach algebra \( B_{\mu}, R \in \mathbb{R}_{>0}^r. \)

Corollary and Definition 11.2 (Impulse response and BIBO stability) For a proper rational function

\[
H = \frac{f(s)}{q(s)} = \sum_{\mu \in \mathbb{N}^r} H_\mu z^\mu \in \mathcal{P} \subset \mathbb{C}(z)
\]

and an input \( u \in \mathcal{D}'_+(\mathbb{R}^r) \) the output

\[
y := H \circ u = H(Y) \ast u = (H \circ \delta) \ast u
\]

is the unique solution of the partial differential equation \( q \circ y = f \circ u \in \mathcal{D}'_+(\mathbb{R}^r) \). For obvious reasons the solution

\[
H \circ \delta = H(Y) = \sum_{\mu \in \mathbb{N}^r} H_\mu \frac{t^\mu S(\mu)}{(\mu S(\mu) - 1 S(\mu))!} \otimes_{\rho \in S(\mu)} Y(t_\rho) \otimes \otimes_{\rho \in S(\mu)} \delta(t_\rho)
\]

is called the fundamental solution or the impulse response of the differential equation. The transfer function \( H \) is called BIBO stable if for a bounded continuous function \( u \) the output \( y := H \circ (uY_\delta) \) with support in \( \mathbb{R}_{\geq 0}^r \) is also bounded.

Remark 11.3 The injectivity and therefore divisibility of the \( \mathbb{C}[s]-\)module \( \mathcal{D}'(\mathbb{R}^r) \) of distributions according to Theorem 3.4 implies that for every rational function \( H = \)
The search for fundamental solutions with good properties is one of the basic tasks of the theory of partial differential equations \[18, 19\].

**Definition and Corollary 11.4**  
1. For \( \mu \in \mathbb{Z}^r \) and \( \alpha \in \mathbb{C}^r \) define \[40, \text{VI.5.13}\]
\[
Y_{\alpha}^{\mu} := e^{\alpha \cdot t} Y_{\mu} \in \mathcal{D}'(\mathbb{R}^r).
\]
Then,
\[
Y_{\alpha}^{\mu} * Y_{\alpha}^{\nu} = Y_{\mu + \nu}^{\alpha}
\]
for \( \mu, \nu \in \mathbb{Z}^r \)
and
\[
(s - \alpha)^{\nu} \circ Y_{\alpha}^{\mu} = Y_{\mu - \nu}^{\alpha}
\]
for \( \mu \in \mathbb{Z}^r, \nu \in \mathbb{N}^r \).

In particular, for \( \mu \in \mathbb{N}^r \) the transfer function
\[
H := (s - \alpha)^{-\mu} = z^\mu (1 - \alpha z)^{-\mu}, \quad (1 - \alpha z)^{-\mu} := \prod_{\rho=1}^{r} (1 - \alpha z^{\rho})^{-\mu^{\rho}},
\]
has the impulse response \( Y_{\alpha}^{\mu} \). If the components of \( \alpha \) have negative real parts the rational function \( (s - \alpha)^{-\mu} \) is proper stable and BIBO stable.

2. The equation
\[
H \circ Y_1 = H(Y) * Y_1 = \sum_{\mu \in \mathbb{N}^r} H_{\mu} Y_{\mu} * Y_1 = \sum_{\mu \in \mathbb{N}^r} H_{\mu} Y_{\mu+1} = \sum_{\mu \in \mathbb{N}^r} H_{\mu} Y_{\mu+1} Y_1
\]
\[
= \hat{H}(t) Y_1
\]
shows that BIBO stability of \( H \) implies the boundedness of \( \hat{H}(t) \) on \( \mathbb{R}_{\geq 0} \).

The preceding corollary and partial fraction decomposition imply the well-known and important fact that each univariate proper stable rational function is BIBO stable.

Sufficient and necessary conditions for BIBO stability can be derived from the disjoint decomposition
\[
\mathbb{N}^r = \biguplus_{S \subseteq \{1, \ldots, r\}} \{ \mu \in \mathbb{N}^r; \text{ supp}(\mu) = S \} = \biguplus_{S} (\{1\} + S + \mathbb{N}^S)
\]
which induces the direct decomposition \( \mathbb{C}[[z]] = \bigoplus_{S} \mathbb{C}[[z_S]] z_S^{1_S} \). Hence, any \( H \in \mathbb{C}[[z]] \) has a unique finite sum representation
\[
H(z) = \sum_{\mu \in \mathbb{N}^r} H_{\mu} z^\mu = \sum_{S \subseteq \{1, \ldots, r\}} H_S(z_S)
\]
with
\[
H_S(z_S) := \sum_{\mu \in \mathbb{N}^r, \text{ supp}(\mu) = S} H_{\mu} z_S^{\mu_S} \in \mathbb{C}[[z_S]].
\]
If $H$ is convergent and $u$ is continuous on $\mathbb{R}^r$ this implies
\[
H(Y) = \sum_S \widetilde{H}_S(t_S) \otimes \rho \in S Y(t_\rho) \otimes \delta(t_\rho)
\]
with
\[
\widetilde{H}_S(t) := (s_1^\frac{1}{t_S} \circ \widetilde{H}_S)(t_S)
\]
\[
= \sum_{\mu \in \mathbb{N}^r, \text{supp}(\mu) = S} H_\mu \frac{\mu_1^{\frac{1}{t_S} - 1}}{(\mu_1 - 1)!} \in \mathcal{O}(\mathbb{C}^S, \exp),
\]
and
\[
(H \circ (uY_1))(t) = (H(Y) * (uY_1))(t) = \sum_S \int_{0}^{t_S} \tilde{H}_S(\tau_S) u(t_S - \tau_S, t_S') d\tau_S.
\]

**Corollary 11.5** A proper rational function $H \in \mathcal{P} \subset \mathbb{C}(z)$ is BIBO stable if the entire functions $\tilde{H}_S(t_S)$, $S \subseteq \{1, \ldots, r\}$, are absolutely integrable over $\mathbb{R}^S_{\geq 0}$. If, in particular,
\[
q(s) = s^d + \sum_{\mu \leq \text{cird}} f_\mu s^\mu, \quad d_\rho > 0 \quad \text{for} \quad \rho = 1, \ldots, r, \quad \text{and} \quad f(s) = \sum_{\mu \leq \text{cird} - 1} f_\mu s^\mu,
\]
the strictly proper rational function
\[
H := \frac{f(s)}{q(s)} = \left( \sum_{\mu \leq \text{cird} - 1} f_\mu z^{d-\mu} \right) \left( \sum_{k=0}^{\infty} (-1)^k \left( \sum_{\mu \leq \text{cird}} q_\mu z^{d-\mu} \right)^k \right)
\]
\[
= \sum_{\mu \geq \text{cird} 1} H_\mu z^\mu = H_{\{1, \ldots, r\}}
\]
is BIBO stable if and only if the function $\tilde{H}(t)Y_1$ with
\[
\tilde{H}(t) := (s_1 \ldots s_r) \circ \tilde{H}(t) := \sum_{\mu \geq \text{cird} 1} H_\mu \frac{\mu_1 - 1}{(\mu - 1)!}
\]
is absolutely integrable, i.e., if
\[
\int_{0}^{\infty} \cdots \int_{0}^{\infty} |\tilde{H}(\tau)| d\tau_1 \cdots d\tau_r < \infty.
\]

With the notations of the preceding corollary we derive integral representations of the functions $\tilde{H}(t)$ and $\tilde{H}(t)$ as in [17, (4.5.7)]. Let $\Gamma_\rho$ with $\rho = 1, \ldots, r$ be simple piecewise smooth closed curves around the origin in $\mathbb{C}$ and $\Gamma := \Gamma_1 \times \cdots \times \Gamma_r$. Then the standard equations
\[
\frac{1}{(2\pi i)^r} \int_{\Gamma} s^\mu ds = \delta_{\mu, -1}, \quad ds := ds_1 \ldots ds_r, \quad \text{for} \quad \mu \in \mathbb{Z}^r, \quad \text{hence}
\]
\[
\frac{\mu_1}{\mu!} = \frac{1}{(2\pi i)^r} \int_{\Gamma} e^{s_1 i} s^{\mu_1 - 1} ds, \quad t \in \mathbb{C}^r, \quad \mu \in \mathbb{N}^r,
\]
hold. Now choose \( R \in \mathbb{R}_{>0} \) and the curves \( \Gamma_{\rho} \) such that the proper rational function \( H \) satisfies

\[
\frac{f(s)}{q(s)} = H(z) = \sum_{\mu \in \mathbb{N}^r} H_{\mu} s^{-\mu} \in B_{R^{-1}}, \quad z := s^{-1}, \quad \text{i.e.,}
\]

\[
\sum_{\mu \in \mathbb{N}^r} |H_{\mu}| R^{-\mu} < \infty, \quad \text{and} \quad \Gamma_{\rho} \subset \{ s_{\rho} \in \mathbb{C} ; |s_{\rho}| \geq R_{\rho} \} \quad \text{for all} \quad \rho.
\]

Corollary 11.6 1. If \( H(s) = \frac{f(s)}{q(s)} = \sum_{\mu \in \mathbb{N}^r} H_{\mu} s^{-\mu} \) is strictly proper as in Corollary 11.5 and if the conditions of (11.1) are satisfied then the integral representations

\[
\tilde{H}(t) = \sum_{\mu \geq e_1} H_{\mu} \frac{\mu^{-1}}{(\mu - 1)!} \int_{\Gamma_1} \cdots \int_{\Gamma_r} e^{t \cdot s} H(s) ds_1 \cdots ds_r
\]

and

\[
\hat{H}(t) = \sum_{\mu \geq e_1} H_{\mu} \frac{\mu^r}{\mu!} \int_{\Gamma_1} \cdots \int_{\Gamma_r} e^{t \cdot s} H(s) ds_1 \cdots ds_r
\]

are valid.

2. If

\[
H(s) = \frac{f(s)}{q(s)} = \sum_{\mu \in \mathbb{N}^r, \supp(\mu) = S} H_{\mu} s^{-\mu}
\]

is an arbitrary proper rational function and if the conditions of (11.1) are satisfied then for each subset \( S \subseteq \{1, \ldots, r\} \) with \(|S|\) elements

\[
\tilde{H}_S(t_S) = \sum_{\mu \in \mathbb{N}^r, \supp(\mu) = S} H_{\mu} \frac{t_S^{\mu - 1_S}}{(\mu_S - 1_S)!}
\]

\[
= \frac{1}{(2\pi i)^{|S|}} \int_{\prod_{\mu \in S} \Gamma_{\mu}} e^{t \cdot s} \left( \sum_{\mu \in \mathbb{N}^r, \supp(\mu) = S} H_{\mu} s^{-\mu} \right) ds.
\]

The preceding integral representation of \( \tilde{H}(t) \) permits to prove Jury’s conjecture. For this purpose let \( H(s) = \frac{f(s)}{q(s)} \in C(s) = C(s_1, s_2) \) be a strictly proper bivariate rational function with \( \deg_{s_2}(f) < \deg_{s_2}(q) \) for \( \rho = 1, 2 \) where \( q(s_1, s_2) \) is a very strict Hurwitz polynomial (VSHP) in the sense of [21, Definition 6]. This signifies that \( q \) is proper and stable with the following additional property. Let

\[
q = s^d + \sum_{\mu \leq e_d} q_{\mu}s^\mu \quad \text{with} \quad q_{\mu} = \sum_{\mu_1=0}^{d_1} b_{\mu_1}^1 (s_2) s_{11}^{\mu_1} \quad \text{and} \quad \sum_{\mu_2=0}^{d_2} b_{\mu_2}^2 (s_1) s_{22}^{\mu_2}
\]

where \( b_{\mu_1}^1 (s_2) \in \mathbb{C}[s_2] \) and \( b_{\mu_2}^2 (s_1) \in \mathbb{C}[s_1] \) are, of course, not zero. The additional condition on \( q \) is that the univariate polynomials \( b_{\mu_1}^1, b_{\mu_2}^2 \) are strict Hurwitz polynomials, i.e., have no zeros in the closed right half-plane of \( \mathbb{C} \).
Theorem 11.7 (Jury’s conjecture) [21, Theorem 23 and Remark] If \( H(s) = \frac{f(s)}{q(s)} \) is a bivariate and strictly proper rational function with a very strict Hurwitz polynomial \( q \) then there is a vector \((a_1, a_2) \in \mathbb{R}^2_+\) and a constant \( C > 0 \) such that
\[
|\tilde{H}(t)| \leq Ce^{-a_1 t_1 - a_2 t_2} \quad \text{for} \quad t \in \mathbb{R}^2_+.
\]
In particular, \( \tilde{H}(t) \) is absolutely integrable over \( \mathbb{R}^2_+ \) and \( H \) is BIBO stable.

Example 11.8 The rational functions
\[
H_1(s) := (s_1 s_2 + s_1 + 2 s_2 + 1)^{-1}
\]
and
\[
H_2(s) := (s_1 s_2 + s_1 + s_2 + 1)^{-1} = (s_1 + 1)^{-1} (s_2 + 1)^{-1}
\]
satisfy the assumptions of the preceding theorem. BIBO stability of \( H_2 \) also follows from Corollary 11.4.

In the future we will study further applications to BIBO stability of Corollaries 11.5 and 11.6, in particular to higher dimensions.

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Bibliography


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